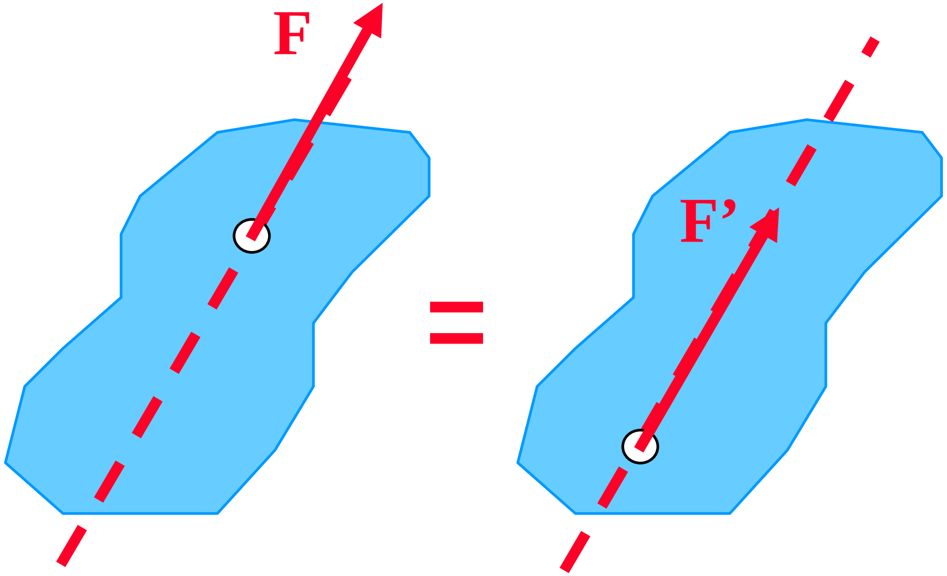
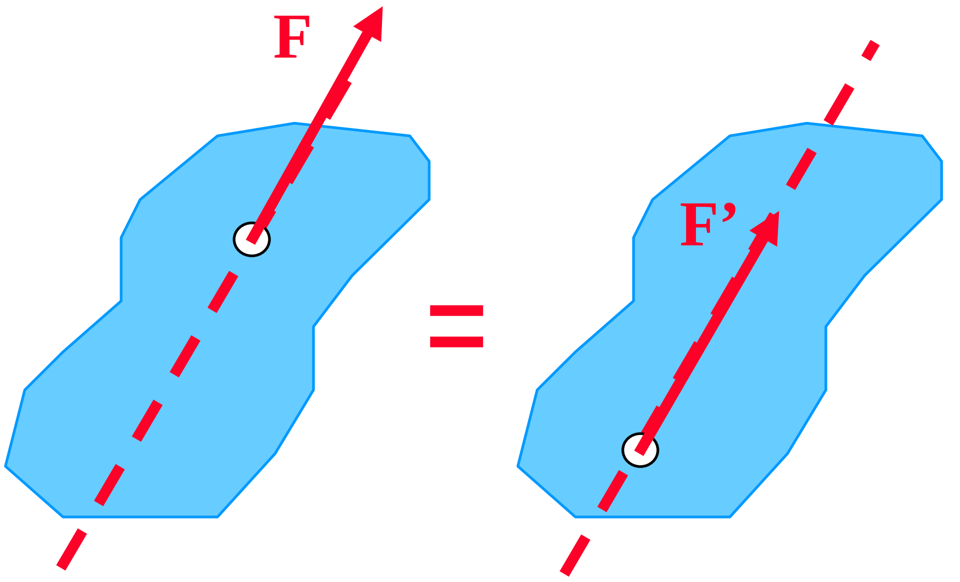
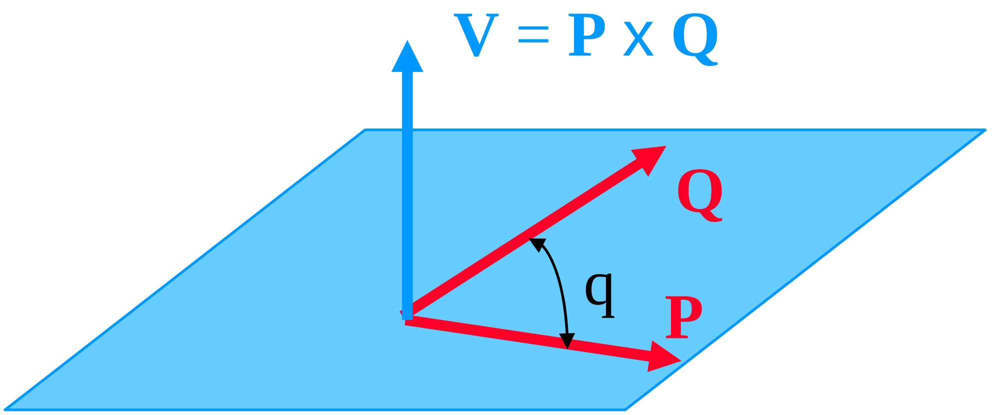
Chapter 3
RIGID BODIES: EQUIVALENT FORCE SYSTEMS



The forces acting on a rigid body can be separated into two groups: (1) ***external forces*** (representing the action of other rigid bodies on the rigid body under consideration) and (2) ***internal forces*** (representing the forces which hold together particles forming the rigid body.



According to the ***principal of transmissibility***, the effect of an ***external*** force on a rigid body remains unchanged if that force is moved along its line of action. Two forces acting on the rigid body at two different points have the same effect on that body if they have the same magnitude, same direction, and same line of action. Two such forces are said to be ***equivalent***.



The ***vector product of two vectors*** is defined as

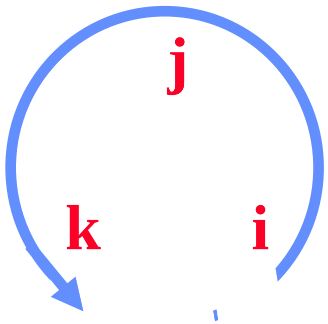
$$\mathbf{V} = \mathbf{P} \times \mathbf{Q}$$

The vector product of **P** and **Q** forms a vector which is perpendicular to both **P** and **Q**, of magnitude

$$V = PQ \sin \theta$$

This vector is directed in such a way that a person located at the tip of **V** observes as counterclockwise the rotation through θ which brings vector **P** in line with vector **Q**. The three vectors **P**, **Q**, and **V** - taken in that order - form a ***right-hand triad***. It follows that

$$\mathbf{Q} \times \mathbf{P} = -(\mathbf{P} \times \mathbf{Q})$$



It follows from the definition of the vector product of two vectors that the vector products of unit vectors **i**, **j**, and **k** are

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}\ ,\ \mathbf{j} \times \mathbf{k} = \mathbf{i}\ ,\ \mathbf{k} \times \mathbf{i} = \mathbf{j}\ ,\ \mathbf{i} \times \mathbf{k} = -\mathbf{j}\ ,\ \mathbf{j} \times \mathbf{i} = -\mathbf{k}\ ,\ \mathbf{k} \times \mathbf{j} = -\mathbf{i}$$

The rectangular components of the vector product **V** of two vectors **P** and **Q** are determined as follows: Given

$$\mathbf{P} = P_x\,\mathbf{i} + P_y\,\mathbf{j} + P_z\,\mathbf{k} \qquad \mathbf{Q} = Q_x\,\mathbf{i} + Q_y\,\mathbf{j} + Q_z\,\mathbf{k}$$

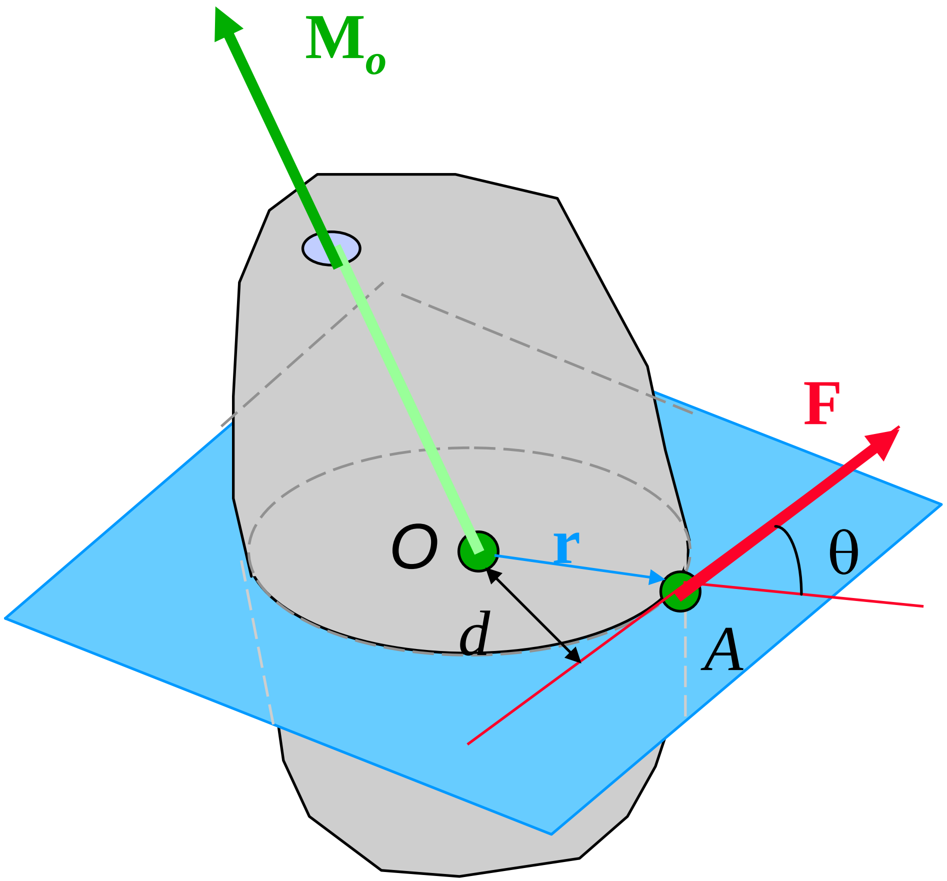
The determinant containing each component of **P** and **Q** is expanded to define the vector **V**, as well as its scalar components

$$\mathbf{P} = P_x\,\mathbf{i} + P_y\,\mathbf{j} + P_z\,\mathbf{k} \qquad \mathbf{Q} = Q_x\,\mathbf{i} + Q_y\,\mathbf{j} + Q_z\,\mathbf{k}$$

$$\mathbf{V} = \mathbf{P} \times \mathbf{Q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} = V_x\,\mathbf{i} + V_y\,\mathbf{j} + V_z\,\mathbf{k}$$

where

$$V_x = P_y\,Q_z - P_z\,Q_y$$
$$V_y = P_z\,Q_x - P_x\,Q_z$$
$$V_z = P_x\,Q_y - P_y\,Q_x$$



The moment of force $\mathbf{F}$ about point O is defined as the vector product

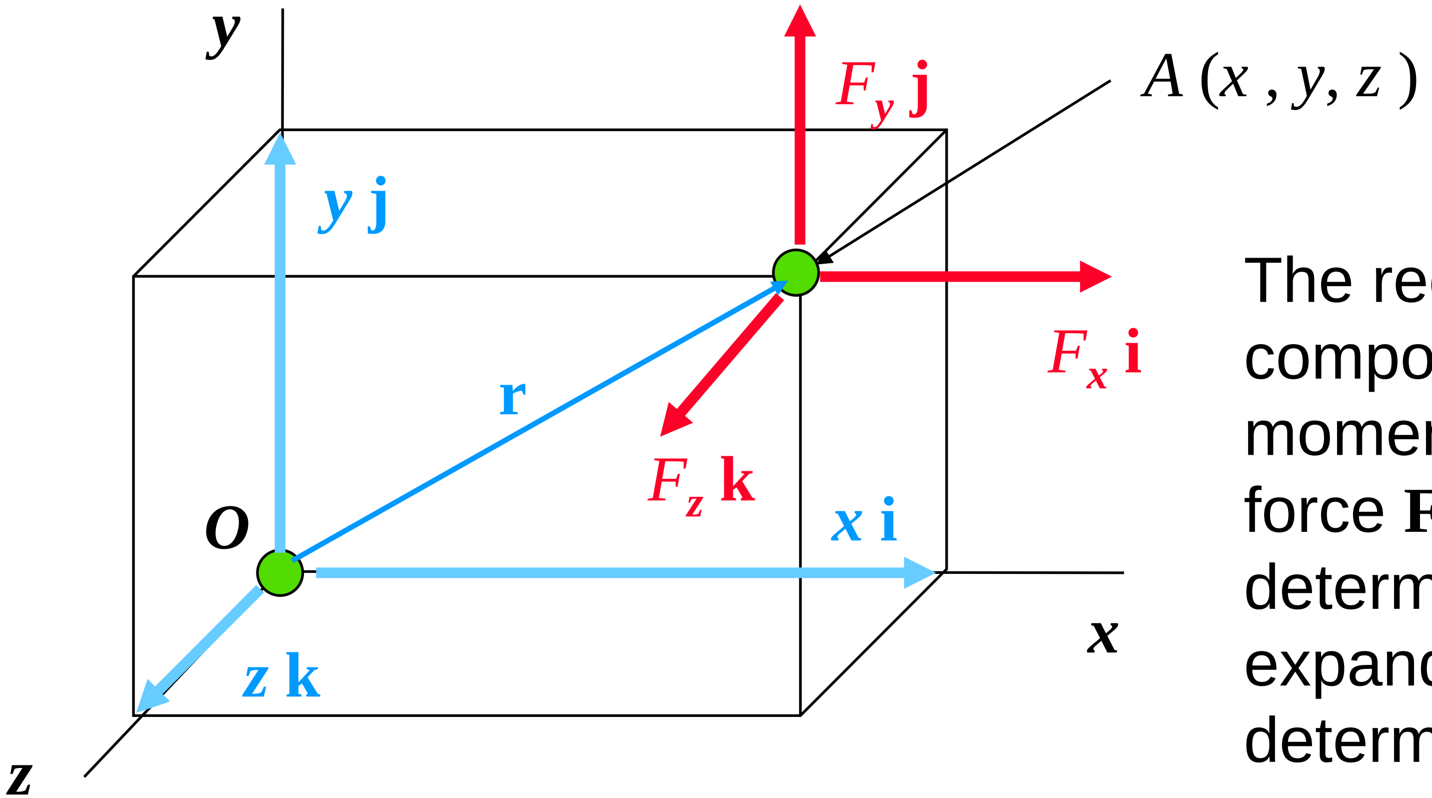
$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$

where $\mathbf{r}$ is the position vector drawn from point O to the point of application of the force $\mathbf{F}$. The angle between the lines of action of $\mathbf{r}$ and $\mathbf{F}$ is θ.

The magnitude of the moment of $\mathbf{F}$ about O can be expressed as

$$M_O = rF \sin\theta = Fd$$

where d is the perpendicular distance from O to the line of action of $\mathbf{F}$.

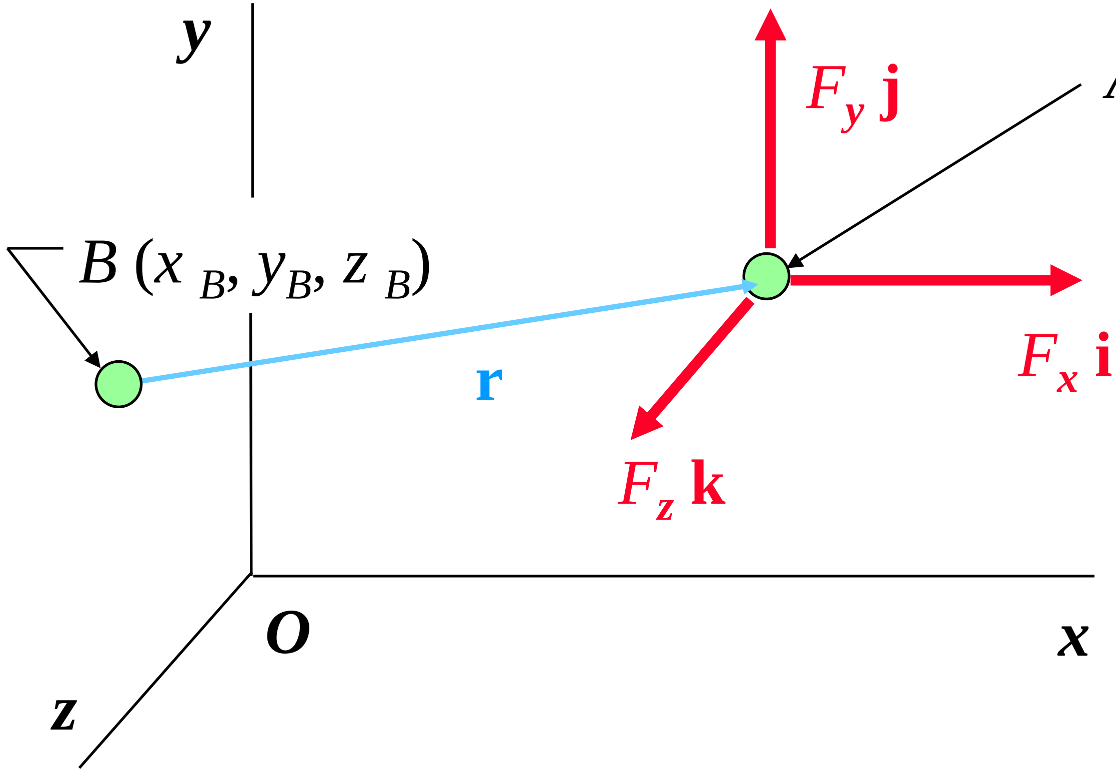


The rectangular components of the moment $\mathbf{M}_o$ of a force $\mathbf{F}$ are determined by expanding the determinant of $\mathbf{r}$ x $\mathbf{F}$.

$$\mathbf{M}_o = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = M_x\,\mathbf{i} + M_y\,\mathbf{j} + M_z\mathbf{k}$$

where $M_x = y\,F_z - z\,F_y \qquad M_y = zF_x - x\,F_z$

$M_z = x\,F_y - y\,F_x$

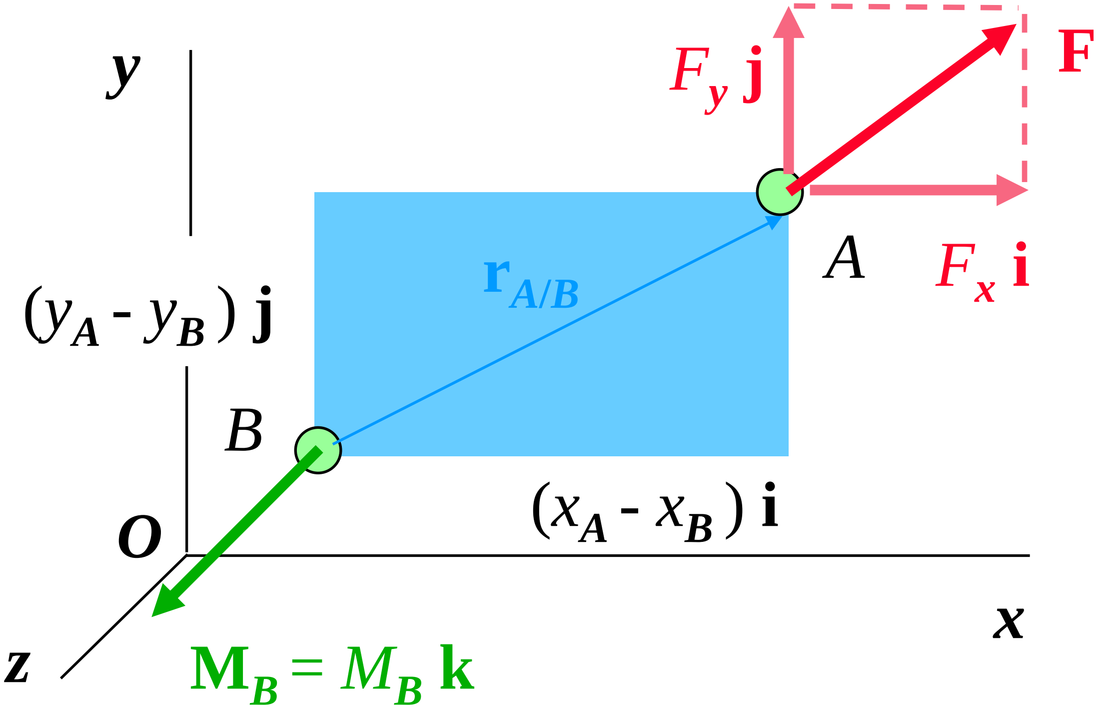


In the more general case of the moment about an arbitrary point B of a force $\mathbf{F}$ applied at A, we have

$$\mathbf{M}_B = \mathbf{r}_{A/B} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_{A/B} & y_{A/B} & z_{A/B} \\ F_x & F_y & F_z \end{vmatrix}$$

where $\mathbf{r}_{A/B} = x_{A/B}\,\mathbf{i} + y_{A/B}\,\mathbf{j} + z_{A/B}\,\mathbf{k}$

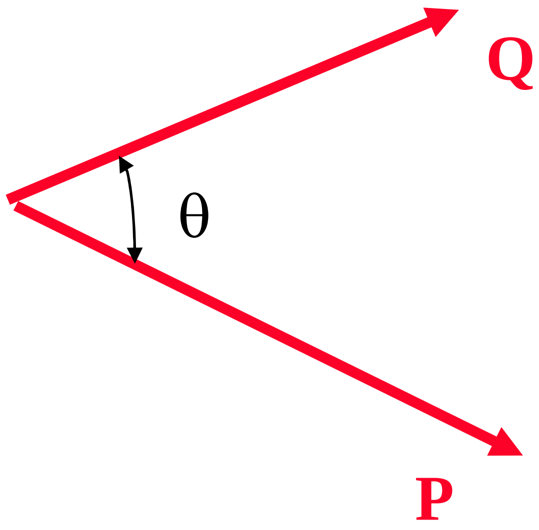
and $x_{A/B} = x_A - x_B \qquad y_{A/B} = y_A - y_B \qquad z_{A/B} = z_A - z_B$



In the case of problems involving only two dimensions, the force **F** can be assumed to lie in the *xy* plane. Its moment about point *B* is perpendicular to that plane. It can be completely defined by the scalar

$$M_B = (x_A - x_B)F_y + (y_A - y_B)\,F_x$$

The ***right-hand rule*** is useful for defining the direction of the moment as either into or out of the plane (positive or negative **k** direction).



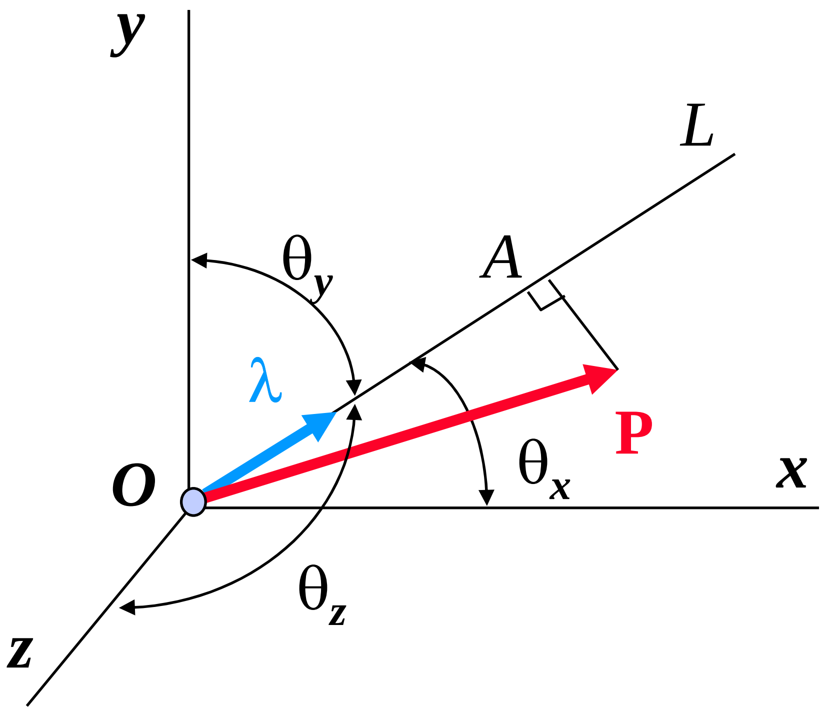
The ***scalar product of two vectors*** **P** and **Q** is denoted as **P•Q** ,and is defined as

$$\mathbf{P} \bullet \mathbf{Q} = PQ \cos\theta$$

where θ is the angle between the two vectors

The scalar product of **P** and **Q** is expressed in terms of the rectangular components of the two vectors as

$$\mathbf{P} \bullet \mathbf{Q} = P_x Q_x + P_y Q_y + P_z Q_z$$



The projection of a vector **P** on an axis *OL* can be obtained by forming the scalar product of **P** and the unit vector λ along *OL*.

$$P_{OL} = \mathbf{P} \bullet \lambda$$

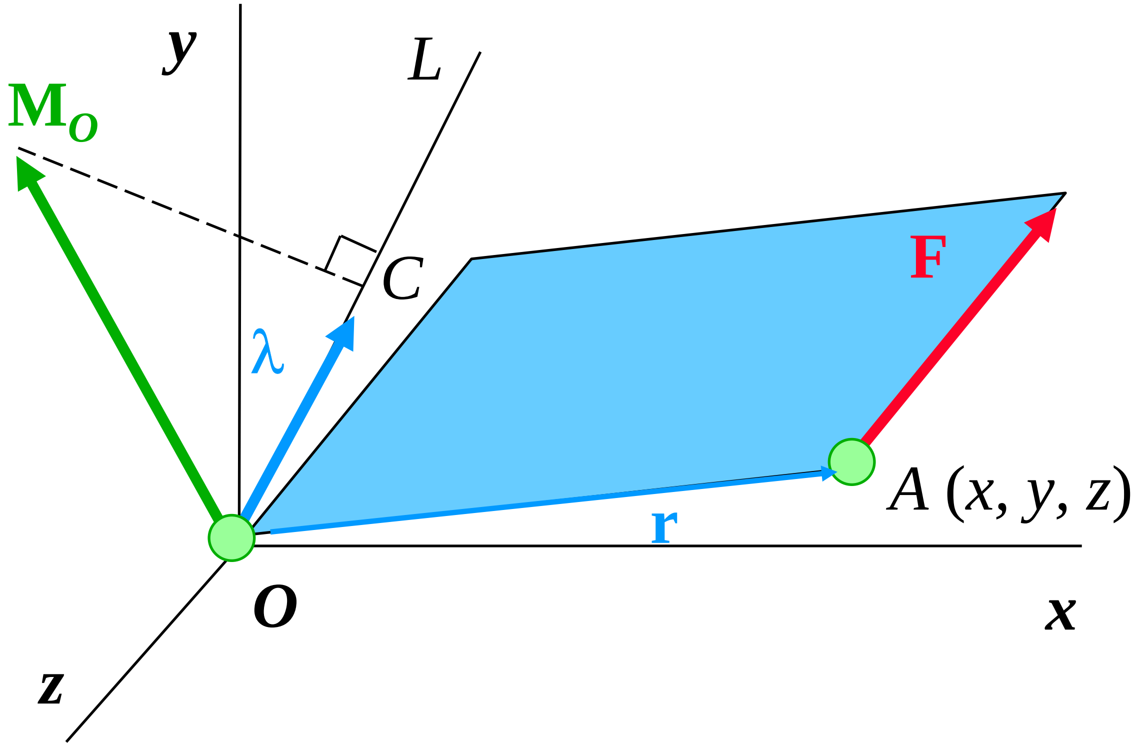
Using rectangular components,

$$P_{OL} = P_x \cos\theta_x + P_y \cos\theta_y + P_z \cos\theta_z$$

The mixed triple product of three vectors **S**, **P**, and **Q** is

$$\mathbf{S} \bullet (\mathbf{P} \times \mathbf{Q}) = \begin{vmatrix} S_x & S_y & S_z \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}$$

The elements of the determinant are the rectangular components of the three vectors.



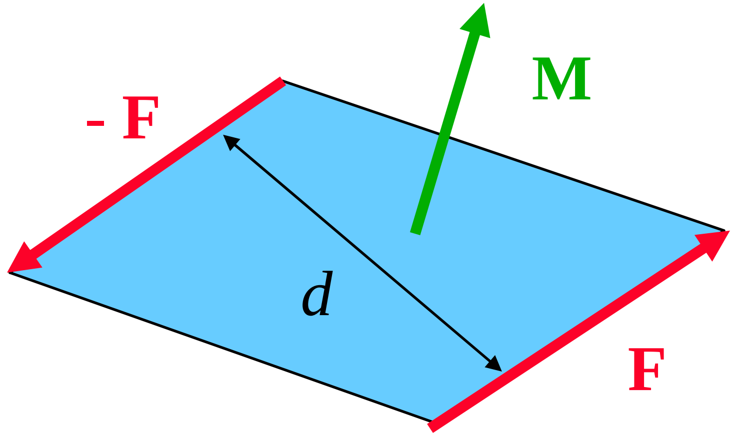
The *moment of a force **F** about an axis* OL is the projection OC on OL of the moment $\mathbf{M}_O$ of the force $\mathbf{F}$. This can be written as a mixed triple product.

$$M_{OL} = \lambda \bullet \mathbf{M}_O = \lambda \bullet (\mathbf{r} \times \mathbf{F}) = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$\lambda_x, \lambda_y, \lambda_z$ = direction cosines of axis OL

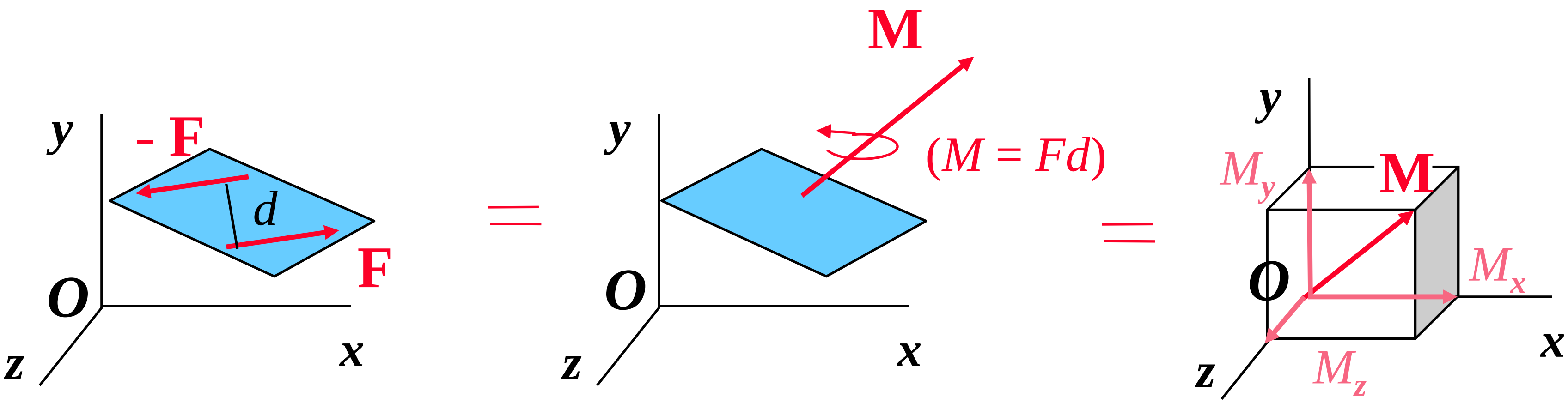
x, y, z = components of $\mathbf{r}$

F_x, F_y, F_z = components of $\mathbf{F}$

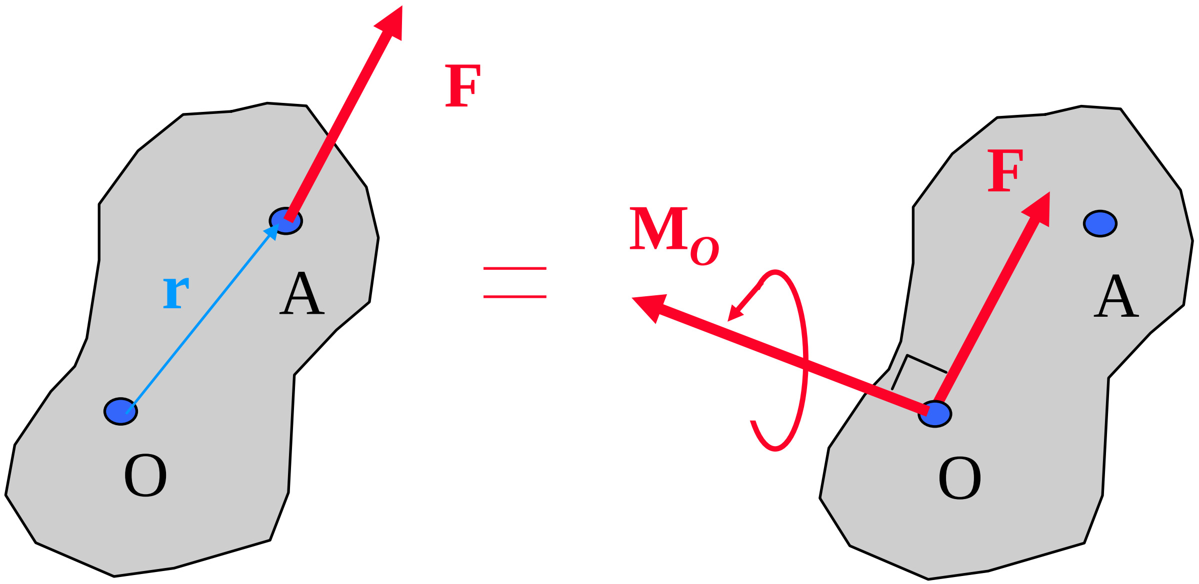


***Two forces* F *and* -F *having the same magnitude, parallel lines of action, and opposite sense are said to form a couple*.**

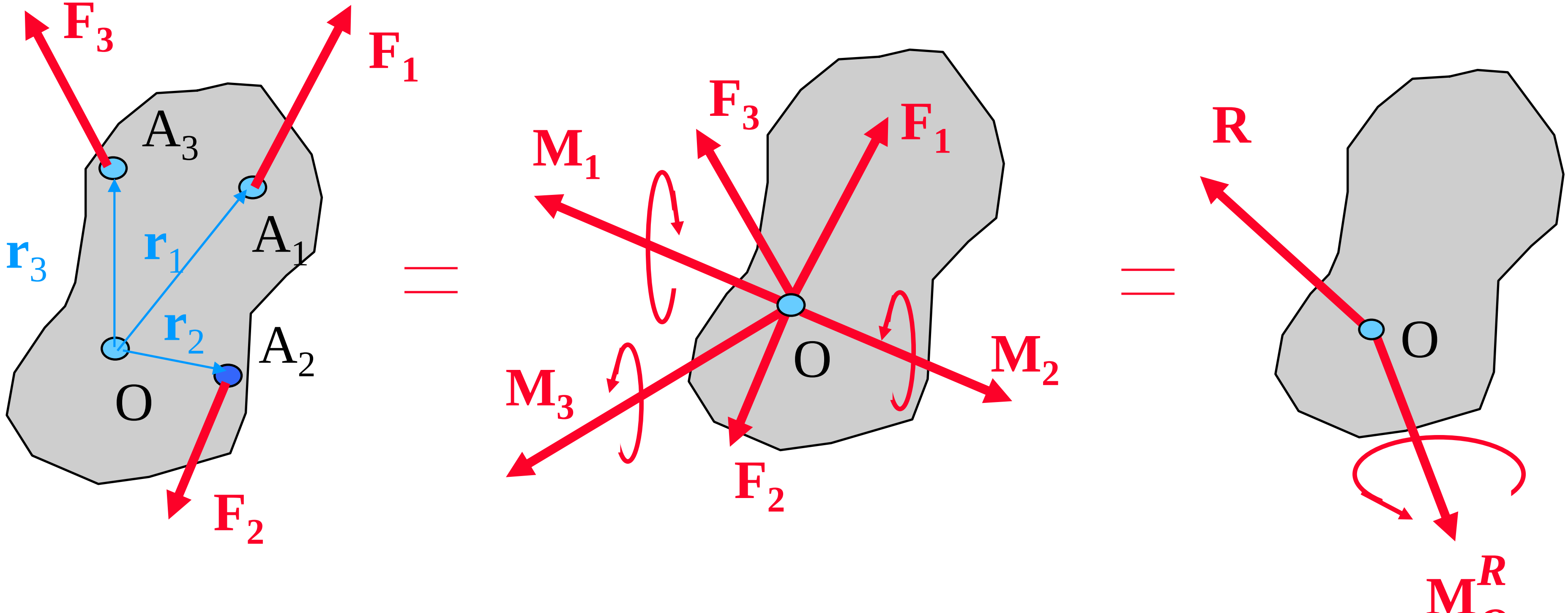
The moment of a couple is independent of the point about which it is computed; it is a vector **M** perpendicular to the plane of the couple and equal in magnitude to the product Fd.



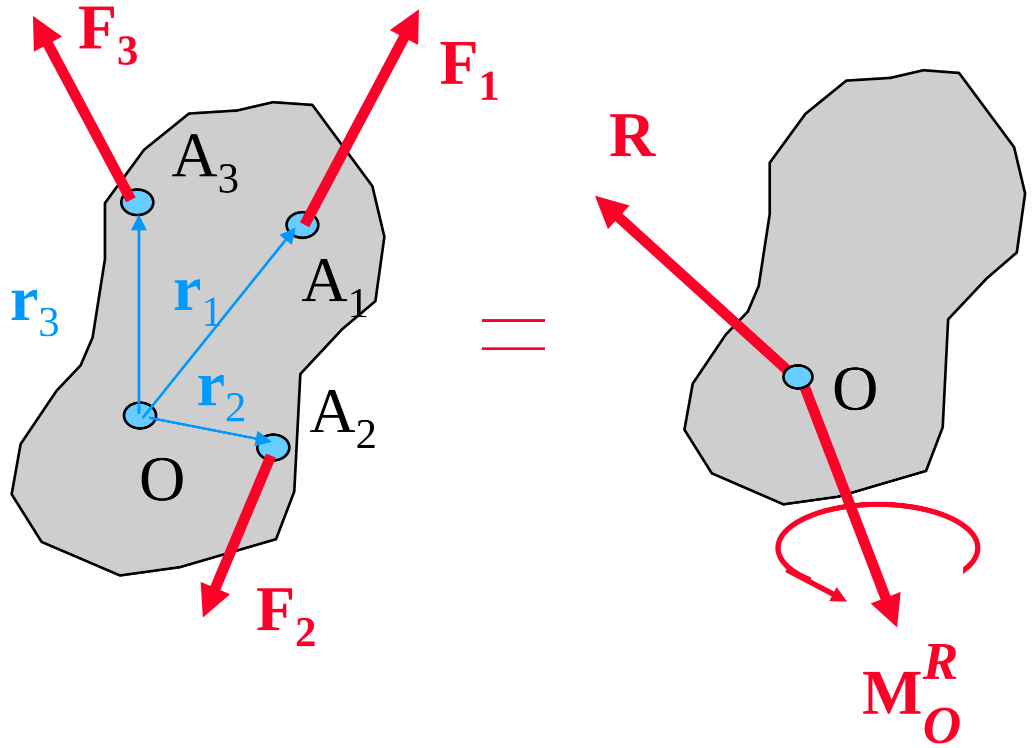
Two couples having the same moment **M** are *equivalent* (they have the same effect on a given rigid body).



Any force **F** acting at a point *A* of a rigid body can be replaced by a ***force-couple system*** at an arbitrary point O, consisting of the force **F** applied at O and a couple of moment $\mathbf{M_O}$ equal to the moment about point O of the force **F** in its original position. The force vector **F** and the couple vector $\mathbf{M_O}$ are always perpendicular to each other.



Any system of forces can be reduced to a force-couple system at a given point O. First, each of the forces of the system is replaced by an equivalent force-couple system at *O*. Then all of the forces are added to obtain a resultant force **R**, and all of couples are added to obtain a resultant couple vector $\mathbf{M}_O^R$. In general, the resultant force **R** and the couple vector $\mathbf{M}_O^R$ will not be perpendicular to each other.

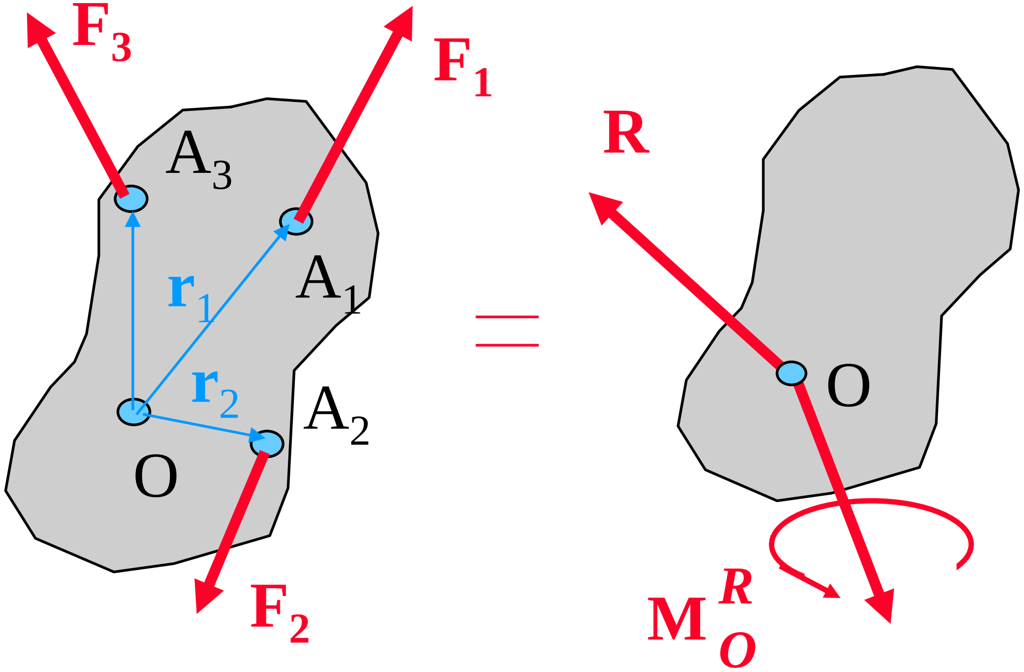


As far as rigid bodies are concerned, ***two systems of forces***, $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ **. . . ,** ***and*** $\mathbf{F'}_1$, $\mathbf{F'}_2$, $\mathbf{F'}_3$ **. . . ,** ***are equivalent if, and only if,***

$$\Sigma\,\mathbf{F} = \Sigma\,\mathbf{F'}$$

and

$$\Sigma\,\mathbf{M_o} = \Sigma\,\mathbf{M_o}'$$



If the resultant force **R** and the resultant couple vector $\mathbf{M}_O^R$ are perpendicular to each other, the force-couple system at *O* can be further reduced to a single resultant force.

This is the case for systems consisting of either
(a) concurrent forces,
(b) coplanar forces, or
(c) parallel forces.

If the resultant force and couple are directed along the same line, the force-couple system is termed a ***wrench***.