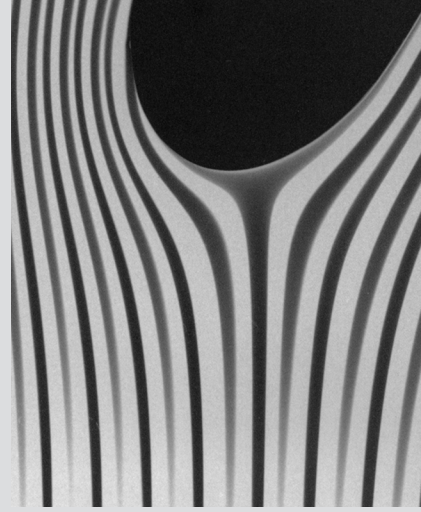


Flow past a blunt body: On any object placed in a moving fluid there is a stagnation point on the front of the object where the velocity is zero. This location has a relatively large pressure and divides the flow field into two portions—one flowing over the body, and one flowing under the body. (Dye in water.) (Photograph by B. R. Munson.)

3

Elementary Fluid Dynamics—The Bernoulli Equation



As was discussed in the previous chapter, there are many situations involving fluids in which the fluid can be considered as stationary. In general, however, the use of fluids involves motion of some type. In fact, a dictionary definition of the word “fluid” is “free to change in form.” In this chapter we investigate some typical fluid motions (fluid dynamics) in an elementary way.

To understand the interesting phenomena associated with fluid motion, one must consider the fundamental laws that govern the motion of fluid particles. Such considerations include the concepts of force and acceleration. We will discuss in some detail the use of Newton’s second law ($\mathbf{F} = m\mathbf{a}$) as it is applied to fluid particle motion that is “ideal” in some sense. We will obtain the celebrated Bernoulli equation and apply it to various flows. Although this equation is one of the oldest in fluid mechanics and the assumptions involved in its derivation are numerous, it can be effectively used to predict and analyze a variety of flow situations. However, if the equation is applied without proper respect for its restrictions, serious errors can arise. Indeed, the Bernoulli equation is appropriately called “the most used and the most abused equation in fluid mechanics.”

The Bernoulli equation may be the most used and abused equation in fluid mechanics.

A thorough understanding of the elementary approach to fluid dynamics involved in this chapter will be useful on its own. It also provides a good foundation for the material in the following chapters where some of the present restrictions are removed and “more nearly exact” results are presented.

3.1 Newton’s Second Law

As a fluid particle moves from one location to another, it usually experiences an acceleration or deceleration. According to Newton’s second law of motion, the net force acting on the fluid particle under consideration must equal its mass times its acceleration,

$$\mathbf{F} = m\mathbf{a}$$

In this chapter we consider the motion of inviscid fluids. That is, the fluid is assumed to have zero viscosity. If the viscosity is zero, then the thermal conductivity of the fluid is also zero and there can be no heat transfer (except by radiation).

In practice there are no inviscid fluids, since every fluid supports shear stresses when it is subjected to a rate of strain displacement. For many flow situations the viscous effects are relatively small compared with other effects. As a first approximation for such cases it is often possible to ignore viscous effects. For example, often the viscous forces developed in flowing water may be several orders of magnitude smaller than forces due to other influences, such as gravity or pressure differences. For other water flow situations, however, the viscous effects may be the dominant ones. Similarly, the viscous effects associated with the flow of a gas are often negligible, although in some circumstances they are very important.

We assume that the fluid motion is governed by pressure and gravity forces only and examine Newton's second law as it applies to a fluid particle in the form:

$$(\text{Net pressure force on a particle}) + (\text{net gravity force on particle}) = (\text{particle mass}) \times (\text{particle acceleration})$$

The results of the interaction between the pressure, gravity, and acceleration provide numerous useful applications in fluid mechanics.

To apply Newton's second law to a fluid (or any other object), we must define an appropriate coordinate system in which to describe the motion. In general the motion will be three-dimensional and unsteady so that three space coordinates and time are needed to describe it. There are numerous coordinate systems available, including the most often used rectangular (x, y, z) and cylindrical (r, θ, z) systems. Usually the specific flow geometry dictates which system would be most appropriate.

In this chapter we will be concerned with two-dimensional motion like that confined to the x - z plane as is shown in Fig. 3.1a. Clearly we could choose to describe the flow in terms of the components of acceleration and forces in the x and z coordinate directions. The resulting equations are frequently referred to as a two-dimensional form of the *Euler equations* of motion in rectangular Cartesian coordinates. This approach will be discussed in Chapter 6.

As is done in the study of dynamics (Ref. 1), the motion of each fluid particle is described in terms of its velocity vector, \mathbf{V} , which is defined as the time rate of change of the position of the particle. The particle's velocity is a vector quantity with a magnitude (the speed, $V = |\mathbf{V}|$) and direction. As the particle moves about, it follows a particular path, the shape of which is governed by the velocity of the particle. The location of the particle along the path is a function of where the particle started at the initial time and its velocity along the path. If it is *steady flow* (i.e., nothing changes with time at a given location in the flow field), each successive particle that passes through a given point [such as point (1) in Fig. 3.1a] will follow the same path. For such cases the path is a fixed line in the x - z plane. Neighboring par-

Inviscid fluid flow in governed by pressure and gravity forces.

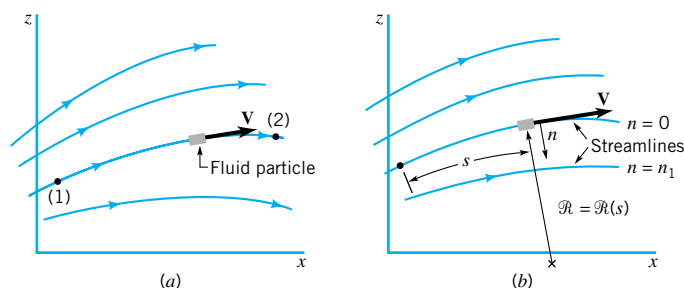


FIGURE 3.1
(a) Flow in the x - z plane.
(b) Flow in terms of streamline and normal coordinates.

ticles that pass on either side of point (1) follow their own paths, which may be of a different shape than the one passing through (1). The entire x - z plane is filled with such paths.

For steady flows each particle slides along its path, and its velocity vector is everywhere tangent to the path. The lines that are tangent to the velocity vectors throughout the flow field are called *streamlines*. For many situations it is easiest to describe the flow in terms of the “streamline” coordinates based on the streamlines as are illustrated in Fig. 3.1*b*. The particle motion is described in terms of its distance, $s = s(t)$, along the streamline from some convenient origin and the local radius of curvature of the streamline, $\mathcal{R} = \mathcal{R}(s)$. The distance along the streamline is related to the particle's speed by $V = ds/dt$, and the radius of curvature is related to shape of the streamline. In addition to the coordinate along the streamline, s , the coordinate normal to the streamline, n , as is shown in Fig. 3.1*b*, will be of use.

To apply Newton's second law to a particle flowing along its streamline, we must write the particle acceleration in terms of the streamline coordinates. By definition, the acceleration is the time rate of change of the velocity of the particle, $\mathbf{a} = d\mathbf{V}/dt$. For two-dimensional flow in the x - z plane, the acceleration has two components—one along the streamline, a_s , the streamwise acceleration, and one normal to the streamline, a_n , the normal acceleration.

The streamwise acceleration results from the fact that the speed of the particle generally varies along the streamline, $V = V(s)$. For example, in Fig. 3.1*a* the speed may be 100 ft/s at point (1) and 50 ft/s at point (2). Thus, by use of the chain rule of differentiation, the s component of the acceleration is given by $a_s = dV/dt = (\partial V/\partial s)(ds/dt) = (\partial V/\partial s)V$. We have used the fact that $V = ds/dt$. The normal component of acceleration, the centrifugal acceleration, is given in terms of the particle speed and the radius of curvature of its path. Thus, $a_n = V^2/\mathcal{R}$, where both V and \mathcal{R} may vary along the streamline. These equations for the acceleration should be familiar from the study of particle motion in physics (Ref. 2) or dynamics (Ref. 1). A more complete derivation and discussion of these topics can be found in **Chapter 4**.

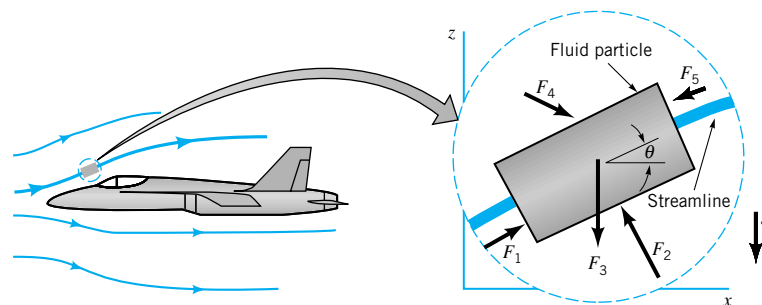
Thus, the components of acceleration in the s and n directions, a_s and a_n , are given by

$$a_s = V \frac{\partial V}{\partial s}, \quad a_n = \frac{V^2}{\mathcal{R}} \quad (3.1)$$

where \mathcal{R} is the local radius of curvature of the streamline, and s is the distance measured along the streamline from some arbitrary initial point. In general there is acceleration along the streamline (because the particle speed changes along its path, $\partial V/\partial s \neq 0$) and acceleration normal to the streamline (because the particle does not flow in a straight line, $\mathcal{R} \neq \infty$). To produce this acceleration there must be a net, nonzero force on the fluid particle.

To determine the forces necessary to produce a given flow (or conversely, what flow results from a given set of forces), we consider the free-body diagram of a small fluid particle as is shown in Fig. 3.2. The particle of interest is removed from its surroundings, and the reactions of the surroundings on the particle are indicated by the appropriate forces

Fluid particles accelerate normal to and along streamlines.



■ **FIGURE 3.2** Isolation of a small fluid particle in a flow field.

present, \mathbf{F}_1 , \mathbf{F}_2 , and so forth. For the present case, the important forces are assumed to be gravity and pressure. Other forces, such as viscous forces and surface tension effects, are assumed negligible. The acceleration of gravity, g , is assumed to be constant and acts vertically, in the negative z direction, at an angle θ relative to the normal to the streamline.

3.2 $\mathbf{F} = m\mathbf{a}$ along a Streamline

Consider the small fluid particle of size δs by δn in the plane of the figure and δy normal to the figure as shown in the free-body diagram of Fig. 3.3. Unit vectors along and normal to the streamline are denoted by $\hat{\mathbf{s}}$ and $\hat{\mathbf{n}}$, respectively. For steady flow, the component of Newton's second law along the streamline direction, s , can be written as

$$\sum \delta F_s = \delta m a_s = \delta m V \frac{\partial V}{\partial s} = \rho \delta \mathcal{V} V \frac{\partial V}{\partial s} \quad (3.2)$$

where $\sum \delta F_s$ represents the sum of the s components of all the forces acting on the particle, which has mass $\delta m = \rho \delta \mathcal{V}$, and $V \partial V / \partial s$ is the acceleration in the s direction. Here, $\delta \mathcal{V} = \delta s \delta n \delta y$ is the particle volume. Equation 3.2 is valid for both compressible and incompressible fluids. That is, the density need not be constant throughout the flow field.

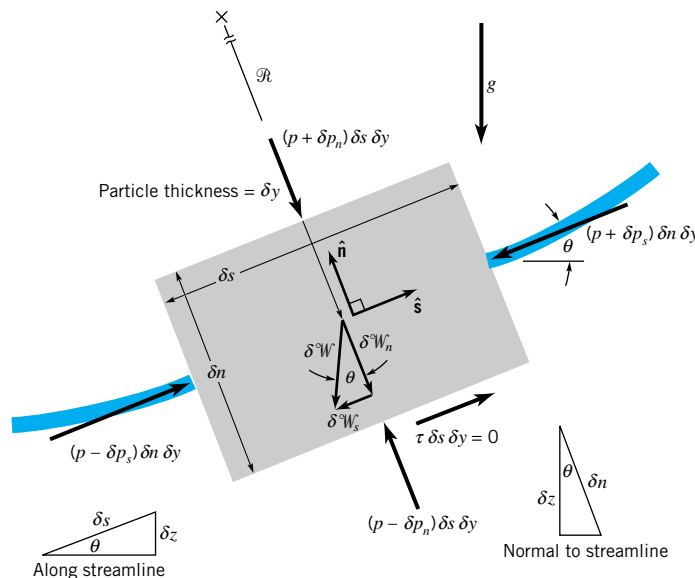
The gravity force (weight) on the particle can be written as $\delta \mathcal{W} = \gamma \delta \mathcal{V}$, where $\gamma = \rho g$ is the specific weight of the fluid (lb/ft³ or N/m³). Hence, the component of the weight force in the direction of the streamline is

$$\delta \mathcal{W}_s = -\delta \mathcal{W} \sin \theta = -\gamma \delta \mathcal{V} \sin \theta$$

If the streamline is horizontal at the point of interest, then $\theta = 0$, and there is no component of particle weight along the streamline to contribute to its acceleration in that direction.

As is indicated in Chapter 2, the pressure is not constant throughout a stationary fluid ($\nabla p \neq 0$) because of the fluid weight. Likewise, in a flowing fluid the pressure is usually not constant. In general, for steady flow, $p = p(s, n)$. If the pressure at the center of the particle shown in Fig. 3.3 is denoted as p , then its average value on the two end faces that are perpendicular to the streamline are $p + \delta p_s$ and $p - \delta p_s$. Since the particle is “small,” we

The component of weight along a streamline depends on the streamline angle.



■ **FIGURE 3.3**
Free-body diagram of a fluid particle for which the important forces are those due to pressure and gravity.

can use a one-term Taylor series expansion for the pressure field (as was done in [Chapter 2](#) for the pressure forces in static fluids) to obtain

$$\delta p_s \approx \frac{\partial p}{\partial s} \frac{\delta s}{2}$$

Thus, if δF_{ps} is the net pressure force on the particle in the streamline direction, it follows that

The net pressure force on a particle is determined by the pressure gradient.

$$\begin{aligned} \delta F_{ps} &= (p - \delta p_s) \delta n \delta y - (p + \delta p_s) \delta n \delta y = -2 \delta p_s \delta n \delta y \\ &= -\frac{\partial p}{\partial s} \delta s \delta n \delta y = -\frac{\partial p}{\partial s} \delta \mathcal{V} \end{aligned}$$

Note that the actual level of the pressure, p , is not important. What produces a net pressure force is the fact that the pressure is not constant throughout the fluid. The nonzero pressure gradient, $\nabla p = \partial p / \partial s \hat{\mathbf{s}} + \partial p / \partial n \hat{\mathbf{n}}$, is what provides a net pressure force on the particle. Viscous forces, represented by $\tau \delta s \delta y$, are zero, since the fluid is inviscid.

Thus, the net force acting in the streamline direction on the particle shown in Fig. 3.3 is given by

$$\sum \delta F_s = \delta \mathcal{W}_s + \delta F_{ps} = \left(-\gamma \sin \theta - \frac{\partial p}{\partial s} \right) \delta \mathcal{V} \quad (3.3)$$

By combining Eqs. 3.2 and 3.3, we obtain the following equation of motion along the streamline direction:

$$-\gamma \sin \theta - \frac{\partial p}{\partial s} = \rho V \frac{\partial V}{\partial s} = \rho a_s \quad (3.4)$$

We have divided out the common particle volume factor, $\delta \mathcal{V}$, that appears in both the force and the acceleration portions of the equation. This is a representation of the fact that it is the fluid density (mass per unit volume), not the mass, per se, of the fluid particle that is important.

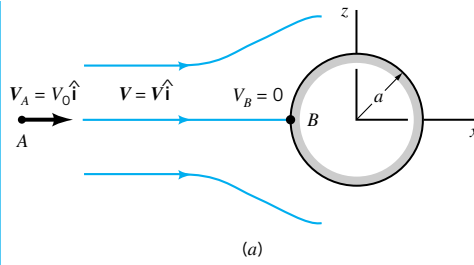
The physical interpretation of Eq. 3.4 is that a change in fluid particle speed is accomplished by the appropriate combination of pressure gradient and particle weight along the streamline. For fluid static situations this balance between pressure and gravity forces is such that no change in particle speed is produced—the right-hand side of Eq. 3.4 is zero, and the particle remains stationary. In a flowing fluid the pressure and weight forces do not necessarily balance—the force unbalance provides the appropriate acceleration and, hence, particle motion.

EXAMPLE 3.1

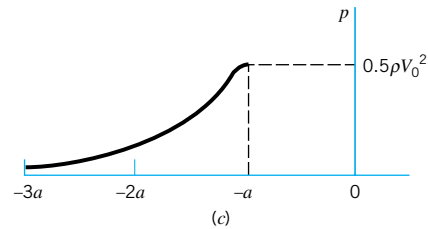
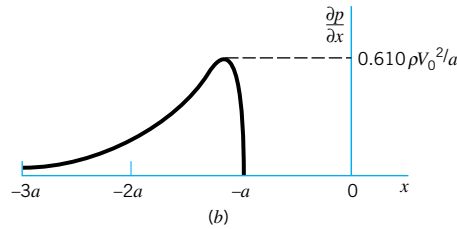
Consider the inviscid, incompressible, steady flow along the horizontal streamline $A-B$ in front of the sphere of radius a , as shown in Fig. E3.1a. From a more advanced theory of flow past a sphere, the fluid velocity along this streamline is

$$V = V_0 \left(1 + \frac{a^3}{x^3} \right)$$

Determine the pressure variation along the streamline from point A far in front of the sphere ($x_A = -\infty$ and $V_A = V_0$) to point B on the sphere ($x_B = -a$ and $V_B = 0$).



■ FIGURE E3.1



SOLUTION

Since the flow is steady and inviscid, Eq. 3.4 is valid. In addition, since the streamline is horizontal, $\sin \theta = \sin 0^\circ = 0$ and the equation of motion along the streamline reduces to

$$\frac{\partial p}{\partial s} = -\rho V \frac{\partial V}{\partial s} \quad (1)$$

With the given velocity variation along the streamline, the acceleration term is

$$V \frac{\partial V}{\partial s} = V \frac{\partial V}{\partial x} = V_0 \left(1 + \frac{a^3}{x^3} \right) \left(-\frac{3V_0 a^3}{x^4} \right) = -3V_0^2 \left(1 + \frac{a^3}{x^3} \right) \frac{a^3}{x^4}$$

where we have replaced s by x since the two coordinates are identical (within an additive constant) along streamline A – B . It follows that $V \partial V / \partial s < 0$ along the streamline. The fluid slows down from V_0 far ahead of the sphere to zero velocity on the “nose” of the sphere ($x = -a$).

Thus, according to Eq. 1, to produce the given motion the pressure gradient along the streamline is

$$\frac{\partial p}{\partial x} = \frac{3\rho a^3 V_0^2 (1 + a^3/x^3)}{x^4} \quad (2)$$

This variation is indicated in Fig. E3.1*b*. It is seen that the pressure increases in the direction of flow ($\partial p / \partial x > 0$) from point A to point B . The maximum pressure gradient ($0.610 \rho V_0^2 / a$) occurs just slightly ahead of the sphere ($x = -1.205a$). It is the pressure gradient that slows the fluid down from $V_A = V_0$ to $V_B = 0$.

The pressure distribution along the streamline can be obtained by integrating Eq. 2 from $p = 0$ (gage) at $x = -\infty$ to pressure p at location x . The result, plotted in Fig. E3.1*c*, is

$$p = -\rho V_0^2 \left[\left(\frac{a}{x} \right)^3 + \frac{(a/x)^6}{2} \right] \quad (\text{Ans})$$

The pressure at B , a stagnation point since $V_B = 0$, is the highest pressure along the streamline ($p_B = \rho V_0^2/2$). As shown in Chapter 9, this excess pressure on the front of the sphere (i.e., $p_B > 0$) contributes to the net drag force on the sphere. Note that the pressure gradient and pressure are directly proportional to the density of the fluid, a representation of the fact that the fluid inertia is proportional to its mass.

Equation 3.4 can be rearranged and integrated as follows. First, we note from Fig. 3.3 that along the streamline $\sin \theta = dz/ds$. Also, we can write $V dV/ds = \frac{1}{2}d(V^2)/ds$. Finally, along the streamline the value of n is constant ($dn = 0$) so that $dp = (\partial p/\partial s) ds + (\partial p/\partial n) dn = (\partial p/\partial s) ds$. Hence, along the streamline $\partial p/\partial s = dp/ds$. These ideas combined with Eq. 3.4 give the following result valid along a streamline

$$-\gamma \frac{dz}{ds} - \frac{dp}{ds} = \frac{1}{2} \rho \frac{d(V^2)}{ds}$$

This simplifies to

$$dp + \frac{1}{2} \rho d(V^2) + \gamma dz = 0 \quad (\text{along a streamline}) \quad (3.5)$$

which can be integrated to give

$$\int \frac{dp}{\rho} + \frac{1}{2} V^2 + gz = C \quad (\text{along a streamline}) \quad (3.6)$$

where C is a constant of integration to be determined by the conditions at some point on the streamline.

In general it is not possible to integrate the pressure term because the density may not be constant and, therefore, cannot be removed from under the integral sign. To carry out this integration we must know specifically how the density varies with pressure. This is not always easily determined. For example, for a perfect gas the density, pressure, and temperature are related according to $p = \rho RT$, where R is the gas constant. To know how the density varies with pressure, we must also know the temperature variation. For now we will assume that the density is constant (incompressible flow). The justification for this assumption and the consequences of compressibility will be considered further in [Section 3.8.1](#) and more fully in [Chapter 11](#).

With the additional assumption that the density remains constant (a very good assumption for liquids and also for gases if the speed is “not too high”), Eq. 3.6 assumes the following simple representation for steady, inviscid, incompressible flow.

$$p + \frac{1}{2} \rho V^2 + \gamma z = \text{constant along streamline} \quad (3.7)$$

The Bernoulli equation can be obtained by integrating $F = ma$ along a streamline.



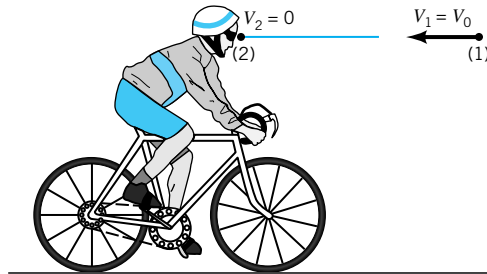
V3.1 Balancing ball

This is the celebrated *Bernoulli equation*—a very powerful tool in fluid mechanics. In 1738 [Daniel Bernoulli](#) (1700–1782) published his *Hydrodynamics* in which an equivalent of this famous equation first appeared. To use it correctly we must constantly remember the basic assumptions used in its derivation: (1) viscous effects are assumed negligible, (2) the flow is assumed to be steady, (3) the flow is assumed to be incompressible, (4) the equation is applicable along a streamline. In the derivation of Eq. 3.7, we assume that the flow takes place in a plane (the x – z plane). In general, this equation is valid for both planar and nonplanar (three-dimensional) flows, provided it is applied along the streamline.

We will provide many examples to illustrate the correct use of the Bernoulli equation and will show how a violation of the basic assumptions used in the derivation of this equation can lead to erroneous conclusions. The constant of integration in the Bernoulli equation can be evaluated if sufficient information about the flow is known at one location along the streamline.

EXAMPLE 3.2

Consider the flow of air around a bicyclist moving through still air with velocity V_0 , as is shown in Fig. E3.2. Determine the difference in the pressure between points (1) and (2).



■ FIGURE E3.2

SOLUTION

In a coordinate system fixed to the bike, it appears as though the air is flowing steadily toward the bicyclist with speed V_0 . If the assumptions of Bernoulli's equation are valid (steady, incompressible, inviscid flow), Eq. 3.7 can be applied as follows along the streamline that passes through (1) and (2)

$$p_1 + \frac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \gamma z_2$$

We consider (1) to be in the free stream so that $V_1 = V_0$ and (2) to be at the tip of the bicyclist's nose and assume that $z_1 = z_2$ and $V_2 = 0$ (both of which, as is discussed in Section 3.4, are reasonable assumptions). It follows that the pressure at (2) is greater than that at (1) by an amount

$$p_2 - p_1 = \frac{1}{2}\rho V_1^2 = \frac{1}{2}\rho V_0^2 \quad (\text{Ans})$$

A similar result was obtained in Example 3.1 by integrating the pressure gradient, which was known because the velocity distribution along the streamline, $V(s)$, was known. The Bernoulli equation is a general integration of $\mathbf{F} = m\mathbf{a}$. To determine $p_2 - p_1$, knowledge of the detailed velocity distribution is not needed—only the “boundary conditions” at (1) and (2) are required. Of course, knowledge of the value of V along the streamline is needed to determine the pressure at points between (1) and (2). Note that if we measure $p_2 - p_1$ we can determine the speed, V_0 . As discussed in Section 3.5, this is the principle upon which many velocity measuring devices are based.

If the bicyclist were accelerating or decelerating, the flow would be unsteady (i.e., $V_0 \neq \text{constant}$) and the above analysis would be incorrect since Eq. 3.7 is restricted to steady flow.

The difference in fluid velocity between two point in a flow field, V_1 and V_2 , can often be controlled by appropriate geometric constraints of the fluid. For example, a garden hose nozzle is designed to give a much higher velocity at the exit of the nozzle than at its entrance where it is attached to the hose. As is shown by the Bernoulli equation, the pressure within

the hose must be larger than that at the exit (for constant elevation, an increase in velocity requires a decrease in pressure if Eq. 3.7 is valid). It is this pressure drop that accelerates the water through the nozzle. Similarly, an airfoil is designed so that the fluid velocity over its upper surface is greater (on the average) than that along its lower surface. From the Bernoulli equation, therefore, the average pressure on the lower surface is greater than that on the upper surface. A net upward force, the lift, results.

3.3 $F = ma$ Normal to a Streamline

In this section we will consider application of Newton's second law in a direction normal to the streamline. In many flows the streamlines are relatively straight, the flow is essentially one-dimensional, and variations in parameters across streamlines (in the normal direction) can often be neglected when compared to the variations along the streamline. However, in numerous other situations valuable information can be obtained from considering $\mathbf{F} = m\mathbf{a}$ normal to the streamlines. For example, the devastating low-pressure region at the center of a tornado can be explained by applying Newton's second law across the nearly circular streamlines of the tornado.

We again consider the force balance on the fluid particle shown in Fig. 3.3. This time, however, we consider components in the normal direction, $\hat{\mathbf{n}}$, and write Newton's second law in this direction as

$$\sum \delta F_n = \frac{\delta m V^2}{\mathcal{R}} = \frac{\rho \delta \mathcal{V} V^2}{\mathcal{R}} \quad (3.8)$$

where $\sum \delta F_n$ represents the sum of n components of all the forces acting on the particle. We assume the flow is steady with a normal acceleration $a_n = V^2/\mathcal{R}$, where \mathcal{R} is the local radius of curvature of the streamlines. This acceleration is produced by the change in direction of the particle's velocity as it moves along a curved path.

We again assume that the only forces of importance are pressure and gravity. The component of the weight (gravity force) in the normal direction is

$$\delta \mathcal{W}_n = -\delta \mathcal{W} \cos \theta = -\gamma \delta \mathcal{V} \cos \theta$$

To apply $F = ma$ normal to streamlines, the normal components of force are needed.

If the streamline is vertical at the point of interest, $\theta = 90^\circ$, and there is no component of the particle weight normal to the direction of flow to contribute to its acceleration in that direction.

If the pressure at the center of the particle is p , then its values on the top and bottom of the particle are $p + \delta p_n$ and $p - \delta p_n$, where $\delta p_n = (\partial p / \partial n)(\delta n/2)$. Thus, if δF_{pn} is the net pressure force on the particle in the normal direction, it follows that

$$\begin{aligned} \delta F_{pn} &= (p - \delta p_n) \delta s \delta y - (p + \delta p_n) \delta s \delta y = -2 \delta p_n \delta s \delta y \\ &= -\frac{\partial p}{\partial n} \delta s \delta n \delta y = -\frac{\partial p}{\partial n} \delta \mathcal{V} \end{aligned}$$

Hence, the net force acting in the normal direction on the particle shown in Fig 3.3 is given by

$$\sum \delta F_n = \delta \mathcal{W}_n + \delta F_{pn} = \left(-\gamma \cos \theta - \frac{\partial p}{\partial n} \right) \delta \mathcal{V} \quad (3.9)$$

By combining Eqs. 3.8 and 3.9 and using the fact that along a line normal to the streamline

$\cos \theta = dz/dn$ (see Fig. 3.3), we obtain the following equation of motion along the normal direction

$$-\gamma \frac{dz}{dn} - \frac{\partial p}{\partial n} = \frac{\rho V^2}{\mathcal{R}} \quad (3.10)$$

Weight and/or pressure can produce curved streamlines.

The physical interpretation of Eq. 3.10 is that a change in the direction of flow of a fluid particle (i.e., a curved path, $\mathcal{R} < \infty$) is accomplished by the appropriate combination of pressure gradient and particle weight normal to the streamline. A larger speed or density or a smaller radius of curvature of the motion requires a larger force unbalance to produce the motion. For example, if gravity is neglected (as is commonly done for gas flows) or if the flow is in a horizontal ($dz/dn = 0$) plane, Eq. 3.10 becomes

$$\frac{\partial p}{\partial n} = -\frac{\rho V^2}{\mathcal{R}}$$

This indicates that the pressure increases with distance away from the center of curvature ($\partial p/\partial n$ is negative since $\rho V^2/\mathcal{R}$ is positive—the positive n direction points toward the “inside” of the curved streamline). Thus, the pressure outside a tornado (typical atmospheric pressure) is larger than it is near the center of the tornado (where an often dangerously low partial vacuum may occur). This pressure difference is needed to balance the centrifugal acceleration associated with the curved streamlines of the fluid motion. (See the photograph at the beginning of Chapter 2.)

EXAMPLE 3.3

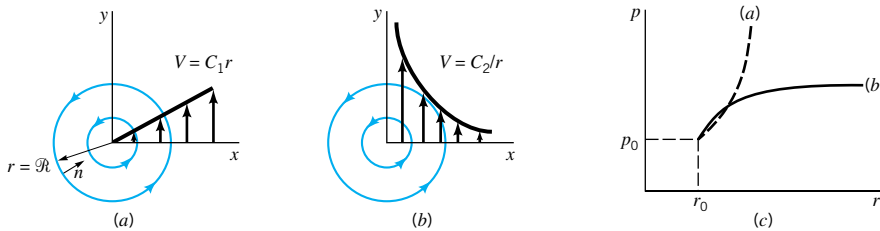
Shown in Figs. E3.3a, b are two flow fields with circular streamlines. The velocity distributions are

$$V(r) = C_1 r \quad \text{for case (a)}$$

and

$$V(r) = \frac{C_2}{r} \quad \text{for case (b)}$$

where C_1 and C_2 are constant. Determine the pressure distributions, $p = p(r)$, for each, given that $p = p_0$ at $r = r_0$.



■ FIGURE E3.3

SOLUTION

We assume the flows are steady, inviscid, and incompressible with streamlines in the horizontal plane ($dz/dn = 0$). Since the streamlines are circles, the coordinate n points in a direction opposite to that of the radial coordinate, $\partial/\partial n = -\partial/\partial r$, and the radius of curvature is given by $\mathcal{R} = r$. Hence, Eq. 3.10 becomes

$$\frac{\partial p}{\partial r} = \frac{\rho V^2}{r}$$

For case (a) this gives

$$\frac{\partial p}{\partial r} = \rho C_1^2 r$$

while for case (b) it gives

$$\frac{\partial p}{\partial r} = \frac{\rho C_2^2}{r^3}$$

For either case the pressure increases as r increases since $\partial p / \partial r > 0$. Integration of these equations with respect to r , starting with a known pressure $p = p_0$ at $r = r_0$, gives

$$p = \frac{1}{2} \rho C_1^2 (r^2 - r_0^2) + p_0 \quad (\text{Ans})$$

for case (a) and

$$p = \frac{1}{2} \rho C_2^2 \left(\frac{1}{r_0^2} - \frac{1}{r^2} \right) + p_0 \quad (\text{Ans})$$

for case (b). These pressure distributions are sketched in Fig. E3.3c. The pressure distributions needed to balance the centrifugal accelerations in cases (a) and (b) are not the same because the velocity distributions are different. In fact for case (a) the pressure increases without bound as $r \rightarrow \infty$, while for case (b) the pressure approaches a finite value as $r \rightarrow \infty$. The streamline patterns are the same for each case, however.

Physically, case (a) represents rigid body rotation (as obtained in a can of water on a turntable after it has been “spun up”) and case (b) represents a free vortex (an approximation of a tornado or the swirl of water in a drain, the “bathtub vortex”). (See the photograph at the beginning of **Chapter 4** for an approximation of this type of flow.)



V3.2 Free vortex

The sum of pressure, elevation, and velocity effects is constant across streamlines.

If we multiply Eq. 3.10 by dn , use the fact that $\partial p / \partial n = dp / dn$ if s is constant, and integrate across the streamline (in the n direction) we obtain

$$\int \frac{dp}{\rho} + \int \frac{V^2}{\mathcal{R}} dn + gz = \text{constant across the streamline} \quad (3.11)$$

To complete the indicated integrations, we must know how the density varies with pressure and how the fluid speed and radius of curvature vary with n . For incompressible flow the density is constant and the integration involving the pressure term gives simply p/ρ . We are still left, however, with the integration of the second term in Eq. 3.11. Without knowing the n dependence in $V = V(s, n)$ and $\mathcal{R} = \mathcal{R}(s, n)$ this integration cannot be completed.

Thus, the final form of Newton’s second law applied across the streamlines for steady, inviscid, incompressible flow is

$$p + \rho \int \frac{V^2}{\mathcal{R}} dn + \gamma z = \text{constant across the streamline} \quad (3.12)$$

As with the Bernoulli equation, we must be careful that the assumptions involved in the derivation of this equation are not violated when it is used.

3.4 Physical Interpretation

In the previous two sections, we developed the basic equations governing fluid motion under a fairly stringent set of restrictions. In spite of the numerous assumptions imposed on these flows, a variety of flows can be readily analyzed with them. A physical interpretation

of the equations will be of help in understanding the processes involved. To this end, we rewrite Eqs. 3.7 and 3.12 here and interpret them physically. Application of $\mathbf{F} = m\mathbf{a}$ along and normal to the streamline results in

$$p + \frac{1}{2}\rho V^2 + \gamma z = \text{constant along the streamline} \quad (3.13)$$

and

$$p + \rho \int \frac{V^2}{\mathcal{R}} dn + \gamma z = \text{constant across the streamline} \quad (3.14)$$

The following basic assumptions were made to obtain these equations: The flow is steady and the fluid is inviscid and incompressible. In practice none of these assumptions is exactly true.

A violation of one or more of the above assumptions is a common cause for obtaining an incorrect match between the “real world” and solutions obtained by use of the Bernoulli equation. Fortunately, many “real-world” situations are adequately modeled by the use of Eqs. 3.13 and 3.14 because the flow is nearly steady and incompressible and the fluid behaves as if it were nearly inviscid.

The Bernoulli equation was obtained by integration of the equation of motion along the “natural” coordinate direction of the streamline. To produce an acceleration, there must be an unbalance of the resultant forces, of which only pressure and gravity were considered to be important. Thus, there are three processes involved in the flow—mass times acceleration (the $\rho V^2/2$ term), pressure (the p term), and weight (the γz term).

Integration of the equation of motion to give Eq. 3.13 actually corresponds to the work-energy principle often used in the study of dynamics [see any standard dynamics text (Ref. 1)]. This principle results from a general integration of the equations of motion for an object in a way very similar to that done for the fluid particle in Section 3.2. With certain assumptions, a statement of the work-energy principle may be written as follows:

The work done on a particle by all forces acting on the particle is equal to the change of the kinetic energy of the particle.

The Bernoulli equation is a mathematical statement of this principle.

As the fluid particle moves, both gravity and pressure forces do work on the particle. Recall that the work done by a force is equal to the product of the distance the particle travels times the component of force in the direction of travel (i.e., work = $\mathbf{F} \cdot \mathbf{d}$). The terms γz and p in Eq. 3.13 are related to the work done by the weight and pressure forces, respectively. The remaining term, $\rho V^2/2$, is obviously related to the kinetic energy of the particle. In fact, an alternate method of deriving the Bernoulli equation is to use the first and second laws of thermodynamics (the energy and entropy equations), rather than Newton’s second law. With the appropriate restrictions, the general energy equation reduces to the Bernoulli equation. This approach is discussed in Section 5.4.

An alternate but equivalent form of the Bernoulli equation is obtained by dividing each term of Eq. 3.7 by the specific weight, γ , to obtain

$$\frac{p}{\gamma} + \frac{V^2}{2g} + z = \text{constant on a streamline}$$

Each of the terms in this equation has the units of energy per weight ($LF/F = L$) or length (feet, meters) and represents a certain type of head.

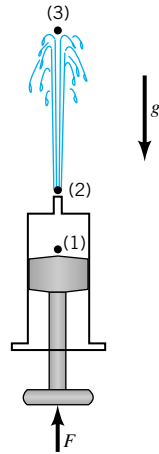
The elevation term, z , is related to the potential energy of the particle and is called the *elevation head*. The pressure term, p/γ , is called the *pressure head* and represents the height of a column of the fluid that is needed to produce the pressure p . The velocity term, $V^2/2g$,

The Bernoulli equation can be written in terms of heights called heads.

is the *velocity head* and represents the vertical distance needed for the fluid to fall freely (neglecting friction) if it is to reach velocity V from rest. The Bernoulli equation states that the sum of the pressure head, the velocity head, and the elevation head is constant along a streamline.

EXAMPLE 3.4

Consider the flow of water from the syringe shown in Fig. E3.4. A force applied to the plunger will produce a pressure greater than atmospheric at point (1) within the syringe. The water flows from the needle, point (2), with relatively high velocity and coasts up to point (3) at the top of its trajectory. Discuss the energy of the fluid at points (1), (2), and (3) by using the Bernoulli equation.



■ FIGURE E3.4

Point	Energy Type		
	Kinetic $\rho V^2/2$	Potential γz	Pressure p
1	Small	Zero	Large
2	Large	Small	Zero
3	Zero	Large	Zero

SOLUTION

If the assumptions (steady, inviscid, incompressible flow) of the Bernoulli equation are approximately valid, it then follows that the flow can be explained in terms of the partition of the total energy of the water. According to Eq. 3.13 the sum of the three types of energy (kinetic, potential, and pressure) or heads (velocity, elevation, and pressure) must remain constant. The following table indicates the relative magnitude of each of these energies at the three points shown in the figure.

The motion results in (or is due to) a change in the magnitude of each type of energy as the fluid flows from one location to another. An alternate way to consider this flow is as follows. The pressure gradient between (1) and (2) produces an acceleration to eject the water from the needle. Gravity acting on the particle between (2) and (3) produces a deceleration to cause the water to come to a momentary stop at the top of its flight.

If friction (viscous) effects were important, there would be an energy loss between (1) and (3) and for the given p_1 the water would not be able to reach the height indicated in the figure. Such friction may arise in the needle (see [Chapter 8](#) on pipe flow) or between the water stream and the surrounding air (see [Chapter 9](#) on external flow).

A net force is required to accelerate any mass. For steady flow the acceleration can be interpreted as arising from two distinct occurrences—a change in speed along the streamline and a change in direction if the streamline is not straight. Integration of the equation of motion along the streamline accounts for the change in speed (kinetic energy change) and results in the Bernoulli equation. Integration of the equation of motion normal to the streamline accounts for the centrifugal acceleration (V^2/\mathcal{R}) and results in Eq. 3.14.

The pressure variation across straight streamlines is hydrostatic.

When a fluid particle travels along a curved path, a net force directed toward the center of curvature is required. Under the assumptions valid for Eq. 3.14, this force may be either gravity or pressure, or a combination of both. In many instances the streamlines are nearly straight ($\mathcal{R} = \infty$) so that centrifugal effects are negligible and the pressure variation across the streamlines is merely hydrostatic (because of gravity alone), even though the fluid is in motion.

EXAMPLE 3.5

Consider the inviscid, incompressible, steady flow shown in Fig. E3.5. From section A to B the streamlines are straight, while from C to D they follow circular paths. Describe the pressure variation between points (1) and (2) and points (3) and (4).

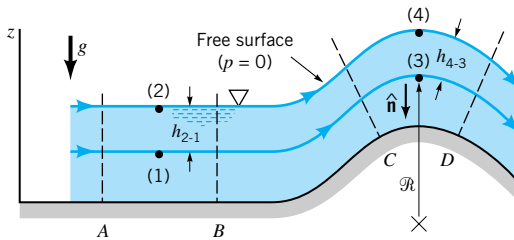


FIGURE E3.5

SOLUTION

With the above assumptions and the fact that $\mathcal{R} = \infty$ for the portion from A to B , Eq. 3.14 becomes

$$p + \gamma z = \text{constant}$$

The constant can be determined by evaluating the known variables at the two locations using $p_2 = 0$ (gage), $z_1 = 0$, and $z_2 = h_{2-1}$ to give

$$p_1 = p_2 + \gamma(z_2 - z_1) = p_2 + \gamma h_{2-1} \quad (\text{Ans})$$

Note that since the radius of curvature of the streamline is infinite, the pressure variation in the vertical direction is the same as if the fluid were stationary.

However, if we apply Eq. 3.14 between points (3) and (4) we obtain (using $dn = -dz$)

$$p_4 + \rho \int_{z_3}^{z_4} \frac{V^2}{\mathcal{R}} (-dz) + \gamma z_4 = p_3 + \gamma z_3$$

With $p_4 = 0$ and $z_4 - z_3 = h_{4-3}$ this becomes

$$p_3 = \gamma h_{4-3} - \rho \int_{z_3}^{z_4} \frac{V^2}{\mathcal{R}} dz \quad (\text{Ans})$$

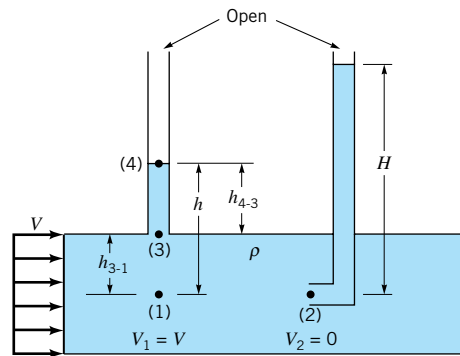
To evaluate the integral, we must know the variation of V and \mathcal{R} with z . Even without this detailed information we note that the integral has a positive value. Thus, the pressure at (3) is less than the hydrostatic value, γh_{4-3} , by an amount equal to $\rho \int_{z_3}^{z_4} (V^2/\mathcal{R}) dz$. This lower pressure, caused by the curved streamline, is necessary to accelerate the fluid around the curved path.

Note that we did not apply the Bernoulli equation (Eq. 3.13) across the streamlines from (1) to (2) or (3) to (4). Rather we used Eq. 3.14. As is discussed in Section 3.8, application of the Bernoulli equation across streamlines (rather than along them) may lead to serious errors.

3.5 Static, Stagnation, Dynamic, and Total Pressure

Each term in the Bernoulli equation can be interpreted as a form of pressure.

A useful concept associated with the Bernoulli equation deals with the stagnation and dynamic pressures. These pressures arise from the conversion of kinetic energy in a flowing fluid into a “pressure rise” as the fluid is brought to rest (as in Example 3.2). In this section we explore various results of this process. Each term of the Bernoulli equation, Eq. 3.13, has the dimensions of force per unit area—psi, lb/ft², N/m². The first term, p , is the actual thermodynamic pressure of the fluid as it flows. To measure its value, one could move along with the fluid, thus being “static” relative to the moving fluid. Hence, it is normally termed the *static pressure*. Another way to measure the static pressure would be to drill a hole in a flat surface and fasten a piezometer tube as indicated by the location of point (3) in Fig. 3.4. As we saw in Example 3.5, the pressure in the flowing fluid at (1) is $p_1 = \gamma h_{3-1} + p_3$, the same as if the fluid were static. From the manometer considerations of Chapter 2, we know that $p_3 = \gamma h_{4-3}$. Thus, since $h_{3-1} + h_{4-3} = h$ it follows that $p_1 = \gamma h$.



■ FIGURE 3.4 Measurement of static and stagnation pressures.

The third term in Eq. 3.13, γz , is termed the *hydrostatic pressure*, in obvious regard to the hydrostatic pressure variation discussed in Chapter 2. It is not actually a pressure but does represent the change in pressure possible due to potential energy variations of the fluid as a result of elevation changes.

The second term in the Bernoulli equation, $\rho V^2/2$, is termed the *dynamic pressure*. Its interpretation can be seen in Fig. 3.4 by considering the pressure at the end of a small tube inserted into the flow and pointing upstream. After the initial transient motion has died out, the liquid will fill the tube to a height of H as shown. The fluid in the tube, including that at its tip, (2), will be stationary. That is, $V_2 = 0$, or point (2) is a *stagnation point*.

If we apply the Bernoulli equation between points (1) and (2), using $V_2 = 0$ and assuming that $z_1 = z_2$, we find that

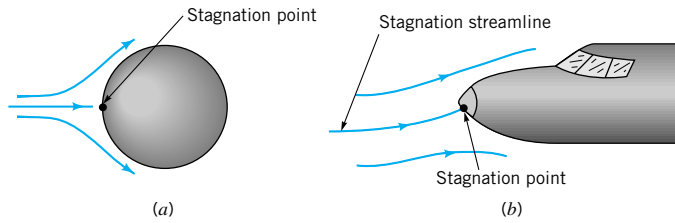
$$p_2 = p_1 + \frac{1}{2}\rho V_1^2$$

Hence, the pressure at the stagnation point is greater than the static pressure, p_1 , by an amount $\rho V_1^2/2$, the dynamic pressure.

It can be shown that there is a stagnation point on any stationary body that is placed into a flowing fluid. Some of the fluid flows “over” and some “under” the object. The dividing line (or surface for two-dimensional flows) is termed the *stagnation streamline* and terminates at the stagnation point on the body. (See the photograph at the beginning of Chapter 3.) For symmetrical objects (such as a sphere) the stagnation point is clearly at the tip or front of the object as shown in Fig. 3.5a. For nonsymmetrical objects such as the airplane shown in Fig. 3.5b, the location of the stagnation point is not always obvious.



V3.3 Stagnation point flow



■ **FIGURE 3.5**
Stagnation points on
bodies in flowing fluids.

If elevation effects are neglected, the *stagnation pressure*, $p + \rho V^2/2$, is the largest pressure obtainable along a given streamline. It represents the conversion of all of the kinetic energy into a pressure rise. The sum of the static pressure, hydrostatic pressure, and dynamic pressure is termed the *total pressure*, p_T . The Bernoulli equation is a statement that the total pressure remains constant along a streamline. That is,

$$p + \frac{1}{2}\rho V^2 + \gamma z = p_T = \text{constant along a streamline} \quad (3.15)$$

Again, we must be careful that the assumptions used in the derivation of this equation are appropriate for the flow being considered.

Knowledge of the values of the static and stagnation pressures in a fluid implies that the fluid speed can be calculated. This is the principle on which the *Pitot-static tube* is based [H. de Pitot (1675–1771)]. As shown in Fig. 3.6, two concentric tubes are attached to two pressure gages (or a differential gage) so that the values of p_3 and p_4 (or the difference $p_3 - p_4$) can be determined. The center tube measures the stagnation pressure at its open tip. If elevation changes are negligible,

$$p_3 = p + \frac{1}{2}\rho V^2$$

where p and V are the pressure and velocity of the fluid upstream of point (2). The outer tube is made with several small holes at an appropriate distance from the tip so that they measure the static pressure. If the elevation difference between (1) and (4) is negligible, then

$$p_4 = p_1 = p$$

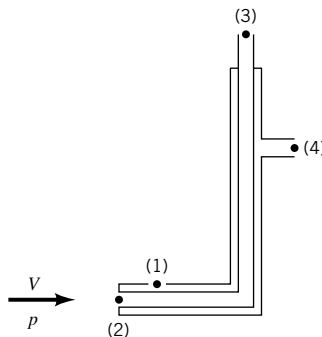
By combining these two equations we see that

$$p_3 - p_4 = \frac{1}{2}\rho V^2$$

which can be rearranged to give

$$V = \sqrt{2(p_3 - p_4)/\rho} \quad (3.16)$$

The actual shape and size of Pitot-static tubes vary considerably. Some of the more common types are shown in Fig. 3.7.

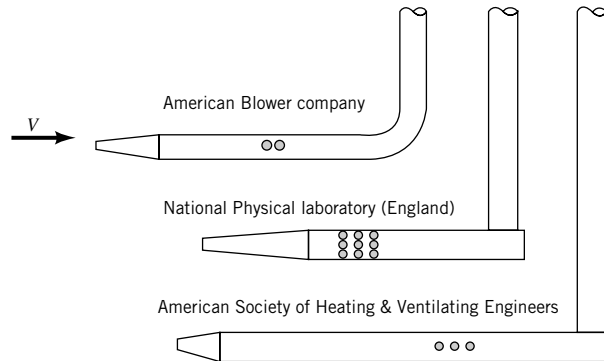


■ **FIGURE 3.6** The Pitot-static tube.



V3.4 Airspeed indicator

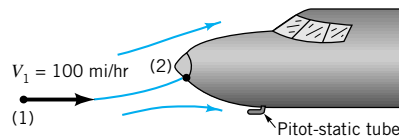
*Pitot-static tubes
measure fluid ve-
locity by converting
velocity into pres-
sure.*



■ **FIGURE 3.7** Typical Pitot-static tube designs.

EXAMPLE 3.6

An airplane flies 100 mi/hr at an elevation of 10,000 ft in a standard atmosphere as shown in Fig. E3.6. Determine the pressure at point (1) far ahead of the airplane, the pressure at the stagnation point on the nose of the airplane, point (2), and the pressure difference indicated by a Pitot-static probe attached to the fuselage.



■ **FIGURE E3.6**

SOLUTION

From **Table C.1** we find that the static pressure at the altitude given is

$$p_1 = 1456 \text{ lb/ft}^2 (\text{abs}) = 10.11 \text{ psia} \quad (\text{Ans})$$

Also, the density is $\rho = 0.001756 \text{ slug/ft}^3$.

If the flow is steady, inviscid, and incompressible and elevation changes are neglected, Eq. 3.13 becomes

$$p_2 = p_1 + \frac{\rho V_1^2}{2}$$

With $V_1 = 100 \text{ mi/hr} = 146.7 \text{ ft/s}$ and $V_2 = 0$ (since the coordinate system is fixed to the airplane) we obtain

$$\begin{aligned} p_2 &= 1456 \text{ lb/ft}^2 + (0.001756 \text{ slugs/ft}^3)(146.7^2 \text{ ft}^2/\text{s}^2)/2 \\ &= (1456 + 18.9) \text{ lb/ft}^2 (\text{abs}) \end{aligned}$$

Hence, in terms of gage pressure

$$p_2 = 18.9 \text{ lb/ft}^2 = 0.1313 \text{ psi} \quad (\text{Ans})$$

Thus, the pressure difference indicated by the Pitot-static tube is

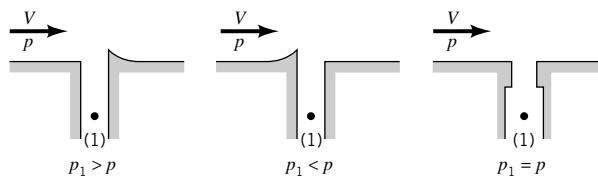
$$p_2 - p_1 = \frac{\rho V_1^2}{2} = 0.1313 \text{ psi} \quad (\text{Ans})$$

Note that it is very easy to obtain incorrect results by using improper units. Do not add lb/in.^2 and lb/ft^2 . Recall that $(\text{slug/ft}^3)(\text{ft}^2/\text{s}^2) = (\text{slug} \cdot \text{ft}/\text{s}^2)/(\text{ft}^2) = \text{lb/ft}^2$.

It was assumed that the flow is incompressible—the density remains constant from (1) to (2). However, since $\rho = p/RT$, a change in pressure (or temperature) will cause a change in density. For this relatively low speed, the ratio of the absolute pressures is nearly unity [i.e., $p_1/p_2 = (10.11 \text{ psia})/(10.11 + 0.1313 \text{ psia}) = 0.987$], so that the density change is negligible. However, at high speed it is necessary to use compressible flow concepts to obtain accurate results. (See [Section 3.8.1](#) and [Chapter 11](#).)

Accurate measurement of static pressure requires great care.

The Pitot-static tube provides a simple, relatively inexpensive way to measure fluid speed. Its use depends on the ability to measure the static and stagnation pressures. Care is needed to obtain these values accurately. For example, an accurate measurement of static pressure requires that none of the fluid's kinetic energy be converted into a pressure rise at the point of measurement. This requires a smooth hole with no burrs or imperfections. As indicated in Fig. 3.8, such imperfections can cause the measured pressure to be greater or less than the actual static pressure.

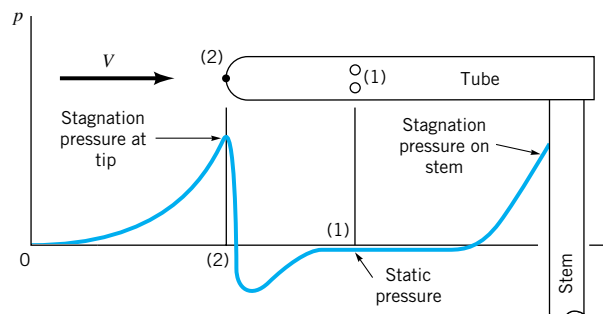


■ **FIGURE 3.8** Incorrect and correct design of static pressure taps.

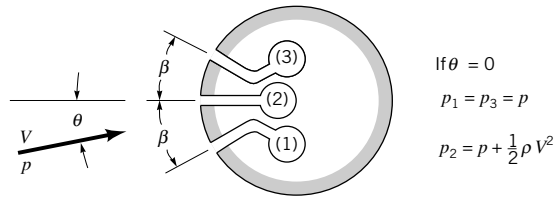
Also, the pressure along the surface of an object varies from the stagnation pressure at its stagnation point to values that may be less than the free stream static pressure. A typical pressure variation for a Pitot-static tube is indicated in Fig. 3.9. Clearly it is important that the pressure taps be properly located to ensure that the pressure measured is actually the static pressure.

In practice it is often difficult to align the Pitot-static tube directly into the flow direction. Any misalignment will produce a nonsymmetrical flow field that may introduce errors. Typically, yaw angles up to 12 to 20° (depending on the particular probe design) give results that are less than 1% in error from the perfectly aligned results. Generally it is more difficult to measure static pressure than stagnation pressure.

One method of determining the flow direction and its speed (thus the velocity) is to use a directional-finding Pitot tube as is illustrated in Fig. 3.10. Three pressure taps are drilled into a small circular cylinder, fitted with small tubes, and connected to three pressure transducers. The cylinder is rotated until the pressures in the two side holes are equal, thus indicating that the center hole points directly upstream. The center tap then measures the stag-



■ **FIGURE 3.9** Typical pressure distribution along a Pitot-static tube.



■ **FIGURE 3.10** Cross section of a directional-finding Pitot-static tube.

Many velocity measuring devices use Pitot-static tube principles.

nation pressure. The two side holes are located at a specific angle ($\beta = 29.5^\circ$) so that they measure the static pressure. The speed is then obtained from $V = [2(p_2 - p_1)/\rho]^{1/2}$.

The above discussion is valid for incompressible flows. At high speeds, compressibility becomes important (the density is not constant) and other phenomena occur. Some of these ideas are discussed in [Section 3.8](#), while others (such as shockwaves for supersonic Pitot-tube applications) are discussed in [Chapter 11](#).

The concepts of static, dynamic, stagnation, and total pressure are useful in a variety of flow problems. These ideas are used more fully in the remainder of the book.

3.6 Examples of Use of the Bernoulli Equation

In this section we illustrate various additional applications of the Bernoulli equation. Between any two points, (1) and (2), on a streamline in steady, inviscid, incompressible flow the Bernoulli equation can be applied in the form

$$p_1 + \frac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \gamma z_2 \quad (3.17)$$

Obviously if five of the six variables are known, the remaining one can be determined. In many instances it is necessary to introduce other equations, such as the conservation of mass. Such considerations will be discussed briefly in this section and in more detail in Chapter 5.

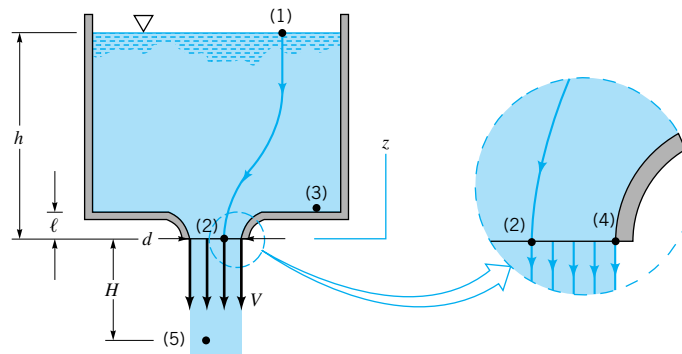
3.6.1 Free Jets

One of the oldest equations in fluid mechanics deals with the flow of a liquid from a large reservoir, as is shown in Fig. 3.11. A jet of liquid of diameter d flows from the nozzle with velocity V as shown. (A nozzle is a device shaped to accelerate a fluid.) Application of Eq. 3.17 between points (1) and (2) on the streamline shown gives

$$\gamma h = \frac{1}{2}\rho V^2$$



V3.5 Flow from a tank



■ **FIGURE 3.11** Vertical flow from a tank.

We have used the facts that $z_1 = h$, $z_2 = 0$, the reservoir is large ($V_1 \cong 0$), open to the atmosphere ($p_1 = 0$ gage), and the fluid leaves as a “free jet” ($p_2 = 0$). Thus, we obtain

$$V = \sqrt{2 \frac{\gamma h}{\rho}} = \sqrt{2gh} \quad (3.18)$$

which is the modern version of a result obtained in 1643 by **Torricelli** (1608–1647), an Italian physicist.

The exit pressure for an incompressible fluid jet is equal to the surrounding pressure.

The fact that the exit pressure equals the surrounding pressure ($p_2 = 0$) can be seen by applying $\mathbf{F} = m\mathbf{a}$, as given by Eq. 3.14, across the streamlines between (2) and (4). If the streamlines at the tip of the nozzle are straight ($\mathcal{R} = \infty$), it follows that $p_2 = p_4$. Since (4) is on the surface of the jet, in contact with the atmosphere, we have $p_4 = 0$. Thus, $p_2 = 0$ also. Since (2) is an arbitrary point in the exit plane of the nozzle, it follows that the pressure is atmospheric across this plane. Physically, since there is no component of the weight force or acceleration in the normal (horizontal) direction, the pressure is constant in that direction.

Once outside the nozzle, the stream continues to fall as a free jet with zero pressure throughout ($p_5 = 0$) and as seen by applying Eq. 3.17 between points (1) and (5), the speed increases according to

$$V = \sqrt{2g(h + H)}$$

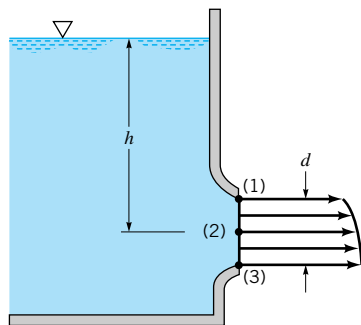
where H is the distance the fluid has fallen outside the nozzle.

Equation 3.18 could also be obtained by writing the Bernoulli equation between points (3) and (4) using the fact that $z_4 = 0$, $z_3 = \ell$. Also, $V_3 = 0$ since it is far from the nozzle, and from hydrostatics, $p_3 = \gamma(h - \ell)$.

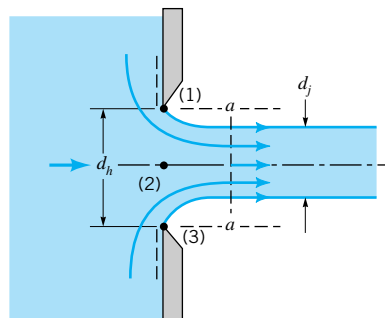
Recall from physics or dynamics that any object dropped from rest through a distance h in a vacuum will obtain the speed $V = \sqrt{2gh}$, the same as the liquid leaving the nozzle. This is consistent with the fact that all of the particle's potential energy is converted to kinetic energy, provided viscous (friction) effects are negligible. In terms of heads, the elevation head at point (1) is converted into the velocity head at point (2). Recall that for the case shown in Fig. 3.11 the pressure is the same (atmospheric) at points (1) and (2).

For the horizontal nozzle of Fig. 3.12, the velocity of the fluid at the centerline, V_2 , will be slightly greater than that at the top, V_1 , and slightly less than that at the bottom, V_3 , due to the differences in elevation. In general, $d \ll h$ and we can safely use the centerline velocity as a reasonable “average velocity.”

If the exit is not a smooth, well-contoured nozzle, but rather a flat plate as shown in Fig. 3.13, the diameter of the jet, d_j , will be less than the diameter of the hole, d_h . This phenomenon, called a *vena contracta* effect, is a result of the inability of the fluid to turn the sharp 90° corner indicated by the dotted lines in the figure.



■ **FIGURE 3.12** Horizontal flow from a tank.

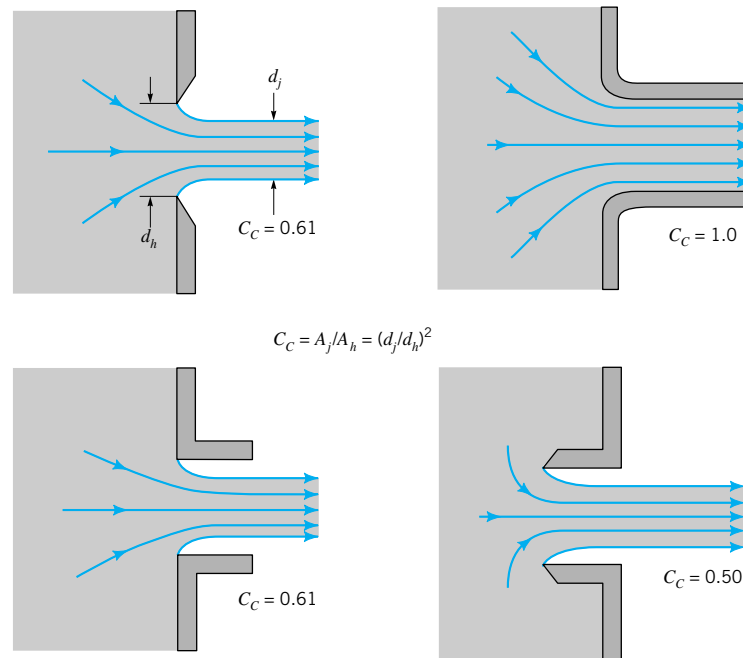


■ **FIGURE 3.13** Vena contracta effect for a sharp-edged orifice.

The diameter of a fluid jet is often smaller than that of the hole from which it flows.

Since the streamlines in the exit plane are curved ($\mathcal{R} < \infty$), the pressure across them is not constant. It would take an infinite pressure gradient across the streamlines to cause the fluid to turn a “sharp” corner ($\mathcal{R} = 0$). The highest pressure occurs along the centerline at (2) and the lowest pressure, $p_1 = p_3 = 0$, is at the edge of the jet. Thus, the assumption of uniform velocity with straight streamlines and constant pressure is not valid at the exit plane. It is valid, however, in the plane of the vena contracta, section $a-a$. The uniform velocity assumption is valid at this section provided $d_j \ll h$, as is discussed for the flow from the nozzle shown in Fig. 3.12.

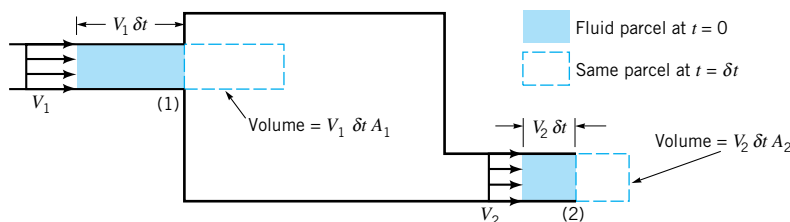
The vena contracta effect is a function of the geometry of the outlet. Some typical configurations are shown in Fig. 3.14 along with typical values of the experimentally obtained contraction coefficient, $C_c = A_j/A_h$, where A_j and A_h are the areas of the jet at the vena contracta and the area of the hole, respectively.



■ **FIGURE 3.14** Typical flow patterns and contraction coefficients for various round exit configurations.

3.6.2 Confined Flows

In many cases the fluid is physically constrained within a device so that its pressure cannot be prescribed a priori as was done for the free jet examples above. Such cases include nozzles and pipes of variable diameter for which the fluid velocity changes because the flow area is different from one section to another. For these situations it is necessary to use the concept of conservation of mass (the continuity equation) along with the Bernoulli equation. The derivation and use of this equation are discussed in detail in [Chapters 4](#) and [5](#). For the needs of this chapter we can use a simplified form of the continuity equation obtained from the following intuitive arguments. Consider a fluid flowing through a fixed volume (such as a tank) that has one inlet and one outlet as shown in Fig. 3.15. If the flow is steady so that there is no additional accumulation of fluid within the volume, the rate at which the fluid flows into the volume must equal the rate at which it flows out of the volume (otherwise, mass would not be conserved).



■ FIGURE 3.15 Steady flow into and out of a tank.

The *mass flowrate* from an outlet, \dot{m} (slugs/s or kg/s), is given by $\dot{m} = \rho Q$, where Q (ft³/s or m³/s) is the *volume flowrate*. If the outlet area is A and the fluid flows across this area (normal to the area) with an average velocity V , then the volume of the fluid crossing this area in a time interval δt is $VA \delta t$, equal to that in a volume of length $V \delta t$ and cross-sectional area A (see Fig. 3.15). Hence, the volume flowrate (volume per unit time) is $Q = VA$. Thus, $\dot{m} = \rho VA$. To conserve mass, the inflow rate must equal the outflow rate. If the inlet is designated as (1) and the outlet as (2), it follows that $\dot{m}_1 = \dot{m}_2$. Thus, conservation of mass requires

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

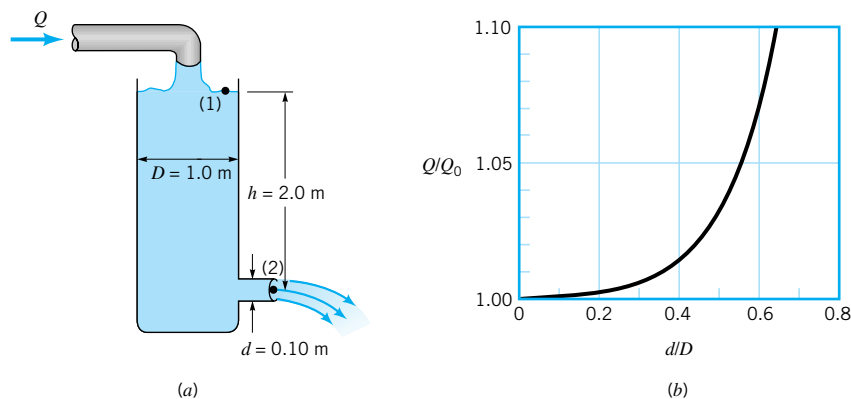
If the density remains constant, then $\rho_1 = \rho_2$, and the above becomes the *continuity equation* for incompressible flow

$$A_1 V_1 = A_2 V_2, \text{ or } Q_1 = Q_2 \quad (3.19)$$

For example, if the outlet flow area is one-half the size of the inlet flow area, it follows that the outlet velocity is twice that of the inlet velocity, since $V_2 = A_1 V_1 / A_2 = 2V_1$. (See the photograph at the beginning of [Chapter 5](#).) The use of the Bernoulli equation and the flowrate equation (continuity equation) is demonstrated by Example 3.7.

EXAMPLE 3.7

A stream of water of diameter $d = 0.1$ m flows steadily from a tank of diameter $D = 1.0$ m as shown in Fig. E3.7a. Determine the flowrate, Q , needed from the inflow pipe if the water depth remains constant, $h = 2.0$ m.



■ FIGURE E3.7

SOLUTION

For steady, inviscid, incompressible flow, the Bernoulli equation applied between points (1) and (2) is

$$p_1 + \frac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \gamma z_2 \quad (1)$$

With the assumptions that $p_1 = p_2 = 0$, $z_1 = h$, and $z_2 = 0$, Eq. 1 becomes

$$\frac{1}{2}V_1^2 + gh = \frac{1}{2}V_2^2 \quad (2)$$

Although the water level remains constant ($h = \text{constant}$), there is an average velocity, V_1 , across section (1) because of the flow from the tank. From Eq. 3.19 for steady incompressible flow, conservation of mass requires $Q_1 = Q_2$, where $Q = AV$. Thus, $A_1V_1 = A_2V_2$, or

$$\frac{\pi}{4}D^2V_1 = \frac{\pi}{4}d^2V_2$$

Hence,

$$V_1 = \left(\frac{d}{D}\right)^2 V_2 \quad (3)$$

Equations 1 and 3 can be combined to give

$$V_2 = \sqrt{\frac{2gh}{1 - (d/D)^4}} = \sqrt{\frac{2(9.81 \text{ m/s}^2)(2.0 \text{ m})}{1 - (0.1 \text{ m}/1 \text{ m})^4}} = 6.26 \text{ m/s}$$

Thus,

$$Q = A_1V_1 = A_2V_2 = \frac{\pi}{4}(0.1 \text{ m})^2(6.26 \text{ m/s}) = 0.0492 \text{ m}^3/\text{s} \quad (\text{Ans})$$

In this example we have not neglected the kinetic energy of the water in the tank ($V_1 \neq 0$). If the tank diameter is large compared to the jet diameter ($D \gg d$), Eq. 3 indicates that $V_1 \ll V_2$ and the assumption that $V_1 \approx 0$ would be reasonable. The error associated with this assumption can be seen by calculating the ratio of the flowrate assuming $V_1 \neq 0$, denoted Q , to that assuming $V_1 = 0$, denoted Q_0 . This ratio, written as

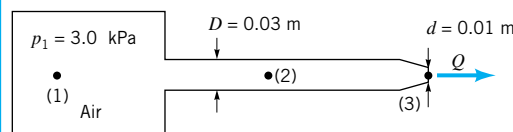
$$\frac{Q}{Q_0} = \frac{V_2}{V_2|_{D=\infty}} = \frac{\sqrt{2gh/[1 - (d/D)^4]}}{\sqrt{2gh}} = \frac{1}{\sqrt{1 - (d/D)^4}}$$

is plotted in Fig. E3.7b. With $0 < d/D < 0.4$ it follows that $1 < Q/Q_0 \leq 1.01$, and the error in assuming $V_1 = 0$ is less than 1%. Thus, it is often reasonable to assume $V_1 = 0$.

The fact that a kinetic energy change is often accompanied by a change in pressure is shown by Example 3.8.

EXAMPLE 3.8

Air flows steadily from a tank, through a hose of diameter $D = 0.03 \text{ m}$ and exits to the atmosphere from a nozzle of diameter $d = 0.01 \text{ m}$ as shown in Fig. E3.8. The pressure in the tank remains constant at 3.0 kPa (gage) and the atmospheric conditions are standard temperature and pressure. Determine the flowrate and the pressure in the hose.



■ FIGURE E3.8

SOLUTION

If the flow is assumed steady, inviscid, and incompressible, we can apply the Bernoulli equation along the streamline shown as

$$\begin{aligned} p_1 + \frac{1}{2}\rho V_1^2 + \gamma z_1 &= p_2 + \frac{1}{2}\rho V_2^2 + \gamma z_2 \\ &= p_3 + \frac{1}{2}\rho V_3^2 + \gamma z_3 \end{aligned}$$

With the assumption that $z_1 = z_2 = z_3$ (horizontal hose), $V_1 = 0$ (large tank), and $p_3 = 0$ (free jet), this becomes

$$V_3 = \sqrt{\frac{2p_1}{\rho}}$$

and

$$p_2 = p_1 - \frac{1}{2}\rho V_2^2 \quad (1)$$

The density of the air in the tank is obtained from the perfect gas law, using standard absolute pressure and temperature, as

$$\begin{aligned} \rho &= \frac{p_1}{RT_1} \\ &= [(3.0 + 101) \text{ kN/m}^2] \\ &\quad \times \frac{10^3 \text{ N/kN}}{(286.9 \text{ N}\cdot\text{m/kg}\cdot\text{K})(15 + 273) \text{ K}} \\ &= 1.26 \text{ kg/m}^3 \end{aligned}$$

Thus, we find that

$$V_3 = \sqrt{\frac{2(3.0 \times 10^3 \text{ N/m}^2)}{1.26 \text{ kg/m}^3}} = 69.0 \text{ m/s}$$

or

$$\begin{aligned} Q &= A_3 V_3 = \frac{\pi}{4} d^2 V_3 = \frac{\pi}{4} (0.01 \text{ m})^2 (69.0 \text{ m/s}) \\ &= 0.00542 \text{ m}^3/\text{s} \end{aligned} \quad (\text{Ans})$$

Note that the value of V_3 is determined strictly by the value of p_1 (and the assumptions involved in the Bernoulli equation), independent of the “shape” of the nozzle. The pressure head within the tank, $p_1/\gamma = (3.0 \text{ kPa})/(9.81 \text{ m/s}^2)(1.26 \text{ kg/m}^3) = 243 \text{ m}$, is converted to the velocity head at the exit, $V_3^2/2g = (69.0 \text{ m/s})^2/(2 \times 9.81 \text{ m/s}^2) = 243 \text{ m}$. Although we used gage pressure in the Bernoulli equation ($p_3 = 0$), we had to use absolute pressure in the perfect gas law when calculating the density.

The pressure within the hose can be obtained from Eq. 1 and the continuity equation (Eq. 3.19)

$$A_2 V_2 = A_3 V_3$$

Hence,

$$V_2 = A_3 V_3 / A_2 = \left(\frac{d}{D}\right)^2 V_3 = \left(\frac{0.01 \text{ m}}{0.03 \text{ m}}\right)^2 (69.0 \text{ m/s}) = 7.67 \text{ m/s}$$

and from Eq. 1

$$\begin{aligned} p_2 &= 3.0 \times 10^3 \text{ N/m}^2 - \frac{1}{2} (1.26 \text{ kg/m}^3)(7.67 \text{ m/s})^2 \\ &= (3000 - 37.1) \text{ N/m}^2 = 2963 \text{ N/m}^2 \end{aligned} \quad (\text{Ans})$$

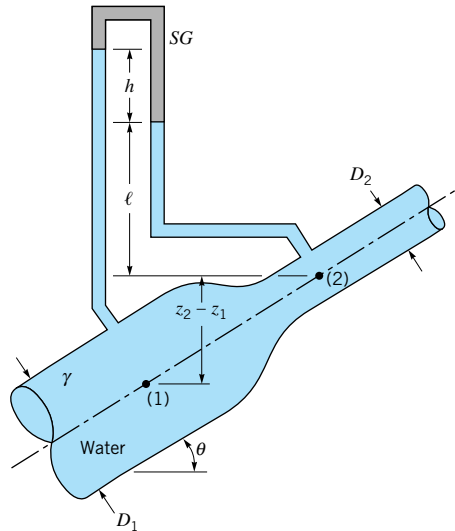
In the absence of viscous effects the pressure throughout the hose is constant and equal to p_2 . Physically, the decreases in pressure from p_1 to p_2 to p_3 accelerate the air and increase its kinetic energy from zero in the tank to an intermediate value in the hose and finally to its maximum value at the nozzle exit. Since the air velocity in the nozzle exit is nine times that in the hose, most of the pressure drop occurs across the nozzle ($p_1 = 3000 \text{ N/m}^2$, $p_2 = 2963 \text{ N/m}^2$ and $p_3 = 0$).

Since the pressure change from (1) to (3) is not too great [that is, in terms of absolute pressure $(p_1 - p_3)/p_1 = 3.0/101 = 0.03$], it follows from the perfect gas law that the density change is also not significant. Hence, the incompressibility assumption is reasonable for this problem. If the tank pressure were considerably larger or if viscous effects were important, the above results would be incorrect.

In many situations the combined effects of kinetic energy, pressure, and gravity are important. Example 3.9 illustrates this.

EXAMPLE 3.9

Water flows through a pipe reducer as is shown in Fig. E3.9. The static pressures at (1) and (2) are measured by the inverted U-tube manometer containing oil of specific gravity, SG , less than one. Determine the manometer reading, h .



■ FIGURE E3.9

SOLUTION

With the assumptions of steady, inviscid, incompressible flow, the Bernoulli equation can be written as

$$p_1 + \frac{1}{2} \rho V_1^2 + \gamma z_1 = p_2 + \frac{1}{2} \rho V_2^2 + \gamma z_2$$

The continuity equation (Eq. 3.19) provides a second relationship between V_1 and V_2 if we

assume the velocity profiles are uniform at those two locations and the fluid incompressible:

$$Q = A_1 V_1 = A_2 V_2$$

By combining these two equations we obtain

$$p_1 - p_2 = \gamma(z_2 - z_1) + \frac{1}{2}\rho V_2^2 [1 - (A_2/A_1)^2] \quad (1)$$

This pressure difference is measured by the manometer and can be determined by using the pressure-depth ideas developed in Chapter 2. Thus,

$$p_1 - \gamma(z_2 - z_1) - \gamma\ell - \gamma h + SG \gamma h + \gamma\ell = p_2$$

or

$$p_1 - p_2 = \gamma(z_2 - z_1) + (1 - SG)\gamma h \quad (2)$$

As discussed in Chapter 2, this pressure difference is neither merely γh nor $\gamma(h + z_1 - z_2)$.

Equations 1 and 2 can be combined to give the desired result as follows:

$$(1 - SG)\gamma h = \frac{1}{2}\rho V_2^2 \left[1 - \left(\frac{A_2}{A_1} \right)^2 \right]$$

or since $V_2 = Q/A_2$

$$h = (Q/A_2)^2 \frac{1 - (A_2/A_1)^2}{2g(1 - SG)} \quad (\text{Ans})$$

The difference in elevation, $z_1 - z_2$, was not needed because the change in elevation term in the Bernoulli equation exactly cancels the elevation term in the manometer equation. However, the pressure difference, $p_1 - p_2$, depends on the angle θ , because of the elevation, $z_1 - z_2$, in Eq. 1. Thus, for a given flowrate, the pressure difference, $p_1 - p_2$, as measured by a pressure gage would vary with θ , but the manometer reading, h , would be independent of θ .



V3.6 Venturi channel

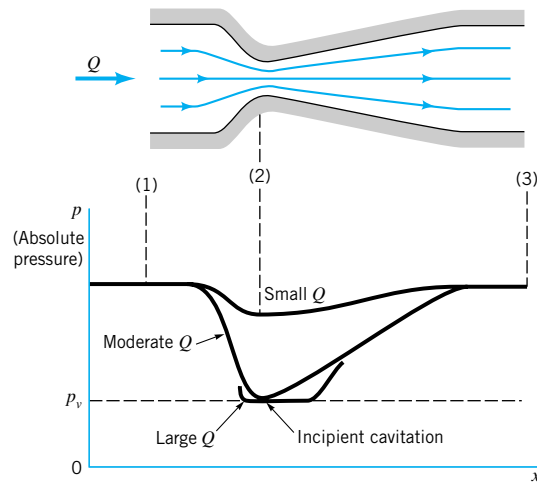
In general, an increase in velocity is accompanied by a decrease in pressure. For example, the velocity of the air flowing over the top surface of an airplane wing is, on the average, faster than that flowing under the bottom surface. Thus, the net pressure force is greater on the bottom than on the top—the wing generates a lift.

If the differences in velocity are considerable, the differences in pressure can also be considerable. For flows of gases, this may introduce compressibility effects as discussed in [Section 3.8](#) and [Chapter 11](#). For flows of liquids, this may result in cavitation, a potentially dangerous situation that results when the liquid pressure is reduced to the vapor pressure and the liquid “boils.”

As discussed in [Chapter 1](#), the vapor pressure, p_v , is the pressure at which vapor bubbles form in a liquid. It is the pressure at which the liquid starts to boil. Obviously this pressure depends on the type of liquid and its temperature. For example, water, which boils at 212 °F at standard atmospheric pressure, 14.7 psia, boils at 80 °F if the pressure is 0.507 psia. That is, $p_v = 0.507$ psia at 80 °F and $p_v = 14.7$ psia at 212 °F. (See [Tables B.1](#) and [B.2](#).)

One way to produce cavitation in a flowing liquid is noted from the Bernoulli equation. If the fluid velocity is increased (for example, by a reduction in flow area as shown in Fig. 3.16) the pressure will decrease. This pressure decrease (needed to accelerate the fluid through the constriction) can be large enough so that the pressure in the liquid is reduced to its vapor pressure. A simple example of cavitation can be demonstrated with an ordinary garden hose. If the hose is “kinked,” a restriction in the flow area in some ways analogous to

Cavitation occurs when the pressure is reduced to the vapor pressure.

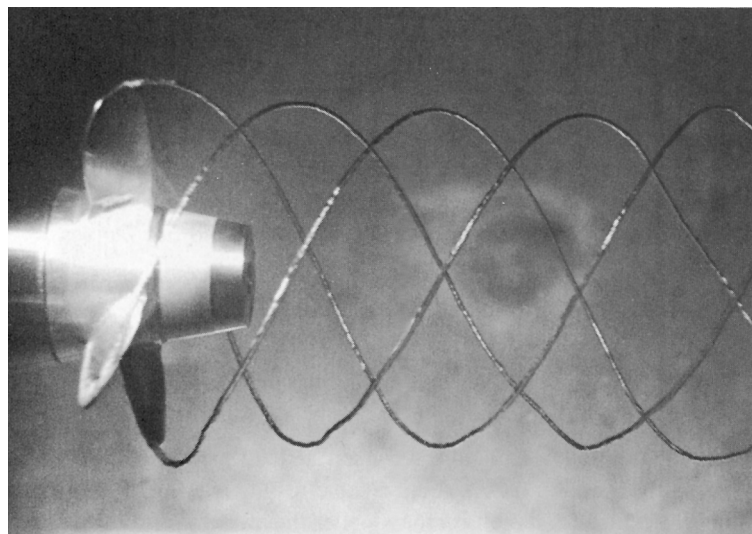


■ **FIGURE 3.16** Pressure variation and cavitation in a variable area pipe.

that shown in Fig. 3.16 will result. The water velocity through this restriction will be relatively large. With a sufficient amount of restriction the sound of the flowing water will change—a definite “hissing” sound is produced. This sound is a result of cavitation.

In such situations boiling occurs (though the temperature need not be high), vapor bubbles form, and then they collapse as the fluid moves into a region of higher pressure (lower velocity). This process can produce dynamic effects (imploding) that cause very large pressure transients in the vicinity of the bubbles. Pressures as large as 100,000 psi (690 MPa) are believed to occur. If the bubbles collapse close to a physical boundary they can, over a period of time, cause damage to the surface in the cavitation area. Tip cavitation from a propeller is shown in Fig. 3.17. In this case the high-speed rotation of the propeller produced a corresponding low pressure on the propeller. Obviously, proper design and use of equipment is needed to eliminate cavitation damage.

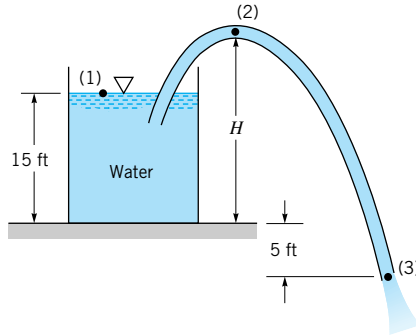
Cavitation can cause damage to equipment.



■ **FIGURE 3.17** Tip cavitation from a propeller. (Photograph courtesy of Garfield Thomas Water Tunnel, Pennsylvania State University.)

EXAMPLE 3.10

Water at 60 °F is siphoned from a large tank through a constant diameter hose as shown in Fig. E3.10. Determine the maximum height of the hill, H , over which the water can be siphoned without cavitation occurring. The end of the siphon is 5 ft below the bottom of the tank. Atmospheric pressure is 14.7 psia.



■ FIGURE E3.10

SOLUTION

If the flow is steady, inviscid, and incompressible we can apply the Bernoulli equation along the streamline from (1) to (2) to (3) as follows:

$$p_1 + \frac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \gamma z_2 = p_3 + \frac{1}{2}\rho V_3^2 + \gamma z_3 \quad (1)$$

With the tank bottom as the datum, we have $z_1 = 15$ ft, $z_2 = H$, and $z_3 = -5$ ft. Also, $V_1 = 0$ (large tank), $p_1 = 0$ (open tank), $p_3 = 0$ (free jet), and from the continuity equation $A_2 V_2 = A_3 V_3$, or because the hose is constant diameter, $V_2 = V_3$. Thus, the speed of the fluid in the hose is determined from Eq. 1 to be

$$\begin{aligned} V_3 &= \sqrt{2g(z_1 - z_3)} = \sqrt{2(32.2 \text{ ft/s}^2)[15 - (-5)]} \text{ ft} \\ &= 35.9 \text{ ft/s} = V_2 \end{aligned}$$

Use of Eq. 1 between points (1) and (2) then gives the pressure p_2 at the top of the hill as

$$p_2 = p_1 + \frac{1}{2}\rho V_1^2 + \gamma z_1 - \frac{1}{2}\rho V_2^2 - \gamma z_2 = \gamma(z_1 - z_2) - \frac{1}{2}\rho V_2^2 \quad (2)$$

From Table B.1, the vapor pressure of water at 60 °F is 0.256 psia. Hence, for incipient cavitation the lowest pressure in the system will be $p = 0.256$ psia. Careful consideration of Eq. 2 and Fig. E3.10 will show that this lowest pressure will occur at the top of the hill. Since we have used gage pressure at point (1) ($p_1 = 0$), we must use gage pressure at point (2) also. Thus, $p_2 = 0.256 - 14.7 = -14.4$ psi and Eq. 2 gives

$$(-14.4 \text{ lb/in.}^2)(144 \text{ in.}^2/\text{ft}^2) = (62.4 \text{ lb/ft}^3)(15 - H)\text{ft} - \frac{1}{2}(1.94 \text{ slugs/ft}^3)(35.9 \text{ ft/s})^2$$

or

$$H = 28.2 \text{ ft} \quad (\text{Ans})$$

For larger values of H , vapor bubbles will form at point (2) and the siphon action may stop.

Note that we could have used absolute pressure throughout ($p_2 = 0.256$ psia and $p_1 = 14.7$ psia) and obtained the same result. The lower the elevation of point (3), the larger the flowrate and, therefore, the smaller the value of H allowed.

We could also have used the Bernoulli equation between (2) and (3), with $V_2 = V_3$, to obtain the same value of H . In this case it would not have been necessary to determine V_2 by use of the Bernoulli equation between (1) and (3).

The above results are independent of the diameter and length of the hose (provided viscous effects are not important). Proper design of the hose (or pipe) is needed to ensure that it will not collapse due to the large pressure difference (vacuum) between the inside and outside of the hose.

3.6.3 Flowrate Measurement

Many types of devices using principles involved in the Bernoulli equation have been developed to measure fluid velocities and flowrates. The Pitot-static tube discussed in Section 3.5 is an example. Other examples discussed below include devices to measure flowrates in pipes and conduits and devices to measure flowrates in open channels. In this chapter we will consider “ideal” flow meters—those devoid of viscous, compressibility, and other “real-world” effects. Corrections for these effects are discussed in Chapters 8 and 10. Our goal here is to understand the basic operating principles of these simple flow meters.

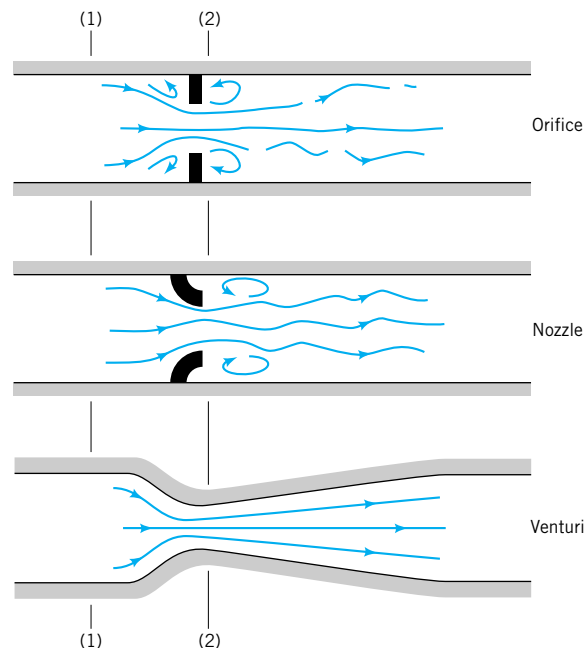
An effective way to measure the flowrate through a pipe is to place some type of restriction within the pipe as shown in Fig. 3.18 and to measure the pressure difference between the low-velocity, high-pressure upstream section (1), and the high-velocity, low-pressure downstream section (2). Three commonly used types of flow meters are illustrated: the *orifice meter*, the *nozzle meter*, and the *Venturi meter*. The operation of each is based on the same physical principles—an increase in velocity causes a decrease in pressure. The difference between them is a matter of cost, accuracy, and how closely their actual operation obeys the idealized flow assumptions.

We assume the flow is horizontal ($z_1 = z_2$), steady, inviscid, and incompressible between points (1) and (2). The Bernoulli equation becomes

$$p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2$$

(The effect of nonhorizontal flow can be incorporated easily by including the change in elevation, $z_1 - z_2$, in the Bernoulli equation.)

Various flow meters are governed by the Bernoulli and continuity equations.



■ **FIGURE 3.18** Typical devices for measuring flowrate in pipes.

If we assume the velocity profiles are uniform at sections (1) and (2), the continuity equation (Eq. 3.19) can be written as

$$Q = A_1 V_1 = A_2 V_2$$

where A_2 is the small ($A_2 < A_1$) flow area at section (2). Combination of these two equations results in the following theoretical flowrate

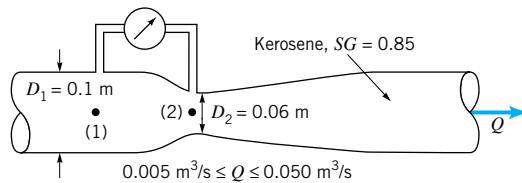
$$Q = A_2 \sqrt{\frac{2(p_1 - p_2)}{\rho[1 - (A_2/A_1)^2]}} \quad (3.20)$$

Thus, for a given flow geometry (A_1 and A_2) the flowrate can be determined if the pressure difference, $p_1 - p_2$, is measured. The actual measured flowrate, Q_{actual} , will be smaller than this theoretical result because of various differences between the “real world” and the assumptions used in the derivation of Eq. 3.20. These differences (which are quite consistent and may be as small as 1 to 2% or as large as 40%, depending on the geometry used) are discussed in [Chapter 8](#).

The flowrate is a function of the pressure difference across the flow meter.

EXAMPLE 3.11

Kerosene ($SG = 0.85$) flows through the Venturi meter shown in Fig. E3.11 with flowrates between 0.005 and $0.050 \text{ m}^3/\text{s}$. Determine the range in pressure difference, $p_1 - p_2$, needed to measure these flowrates.



■ FIGURE E3.11

SOLUTION

If the flow is assumed to be steady, inviscid, and incompressible, the relationship between flowrate and pressure is given by Eq. 3.20. This can be rearranged to give

$$p_1 - p_2 = \frac{Q^2 \rho [1 - (A_2/A_1)^2]}{2 A_2^2}$$

With a density of the flowing fluid of

$$\rho = SG \rho_{\text{H}_2\text{O}} = 0.85(1000 \text{ kg/m}^3) = 850 \text{ kg/m}^3$$

the pressure difference for the smallest flowrate is

$$\begin{aligned} p_1 - p_2 &= (0.005 \text{ m}^3/\text{s})^2 (850 \text{ kg/m}^3) \frac{[1 - (0.06 \text{ m}/0.10 \text{ m})^4]}{2 [(\pi/4)(0.06 \text{ m})^2]^2} \\ &= 1160 \text{ N/m}^2 = 1.16 \text{ kPa} \end{aligned}$$

Likewise, the pressure difference for the largest flowrate is

$$\begin{aligned} p_1 - p_2 &= (0.05)^2 (850) \frac{[1 - (0.06/0.10)^4]}{2 [(\pi/4)(0.06)^2]^2} \\ &= 1.16 \cdot 10^5 \text{ N/m}^2 = 116 \text{ kPa} \end{aligned}$$

Thus,

$$1.16 \text{ kPa} \leq p_1 - p_2 \leq 116 \text{ kPa} \quad (\text{Ans})$$

These values represent the pressure differences for inviscid, steady, incompressible conditions. The ideal results presented here are independent of the particular flow meter geometry—an orifice, nozzle, or Venturi meter (see Fig. 3.18).

It is seen from Eq. 3.20 that the flowrate varies as the square root of the pressure difference. Hence, as indicated by the numerical results, a tenfold increase in flowrate requires a one-hundredfold increase in pressure difference. This nonlinear relationship can cause difficulties when measuring flowrates over a wide range of values. Such measurements would require pressure transducers with a wide range of operation. An alternative is to use two flow meters in parallel—one for the larger and one for the smaller flowrate ranges.

Other flow meters based on the Bernoulli equation are used to measure flowrates in open channels such as flumes and irrigation ditches. Two of these devices, the *sluice gate* and the *sharp-crested weir*, are discussed below under the assumption of steady, inviscid, incompressible flow. These and other open-channel flow devices are discussed in more detail in Chapter 10.

The sluice gate as shown in Fig. 3.19 is often used to regulate and measure the flowrate in an open channel. The flowrate, Q , is a function of the water depth upstream, z_1 , the width of the gate, b , and the gate opening, a . Application of the Bernoulli equation and continuity equation between points (1) and (2) can provide a good approximation to the actual flowrate obtained. We assume the velocity profiles are uniform sufficiently far upstream and downstream of the gate.

Thus, we apply the Bernoulli and continuity equations between points on the free surfaces at (1) and (2) to give

$$p_1 + \frac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \gamma z_2$$

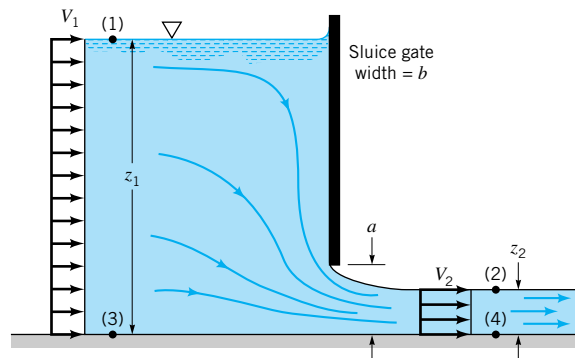
and

$$Q = A_1 V_1 = b V_1 z_1 = A_2 V_2 = b V_2 z_2$$

With the fact that $p_1 = p_2 = 0$, these equations can be combined and rearranged to give the flowrate as

$$Q = z_2 b \sqrt{\frac{2g(z_1 - z_2)}{1 - (z_2/z_1)^2}} \quad (3.21)$$

The flowrate under a sluice gate depends on the water depths on either side of the gate.



■ FIGURE 3.19 Sluice gate geometry.

In the limit of $z_1 \gg z_2$ this result simply becomes

$$Q = z_2 b \sqrt{2gz_1}$$

This limiting result represents the fact that if the depth ratio, z_1/z_2 , is large, the kinetic energy of the fluid upstream of the gate is negligible and the fluid velocity after it has fallen a distance $(z_1 - z_2) \approx z_1$ is approximately $V_2 = \sqrt{2gz_1}$.

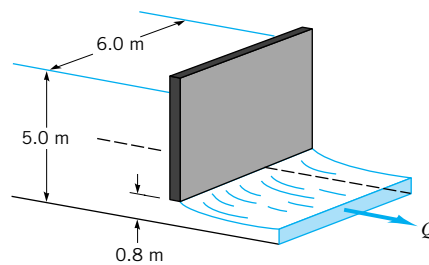
The results of Eq. 3.21 could also be obtained by using the Bernoulli equation between points (3) and (4) and the fact that $p_3 = \gamma z_1$ and $p_4 = \gamma z_2$ since the streamlines at these sections are straight. In this formulation, rather than the potential energies at (1) and (2), we have the pressure contributions at (3) and (4).

The downstream depth, z_2 , not the gate opening, a , was used to obtain the result of Eq. 3.21. As was discussed relative to flow from an orifice (Fig. 3.14), the fluid cannot turn a sharp 90° corner. A vena contracta results with a contraction coefficient, $C_c = z_2/a$, less than 1. Typically C_c is approximately 0.61 over the depth ratio range of $0 < a/z_1 < 0.2$. For larger values of a/z_1 the value of C_c increases rapidly.

A vena contracta occurs as water flows under a sluice gate.

EXAMPLE 3.12

Water flows under the sluice gate shown in Fig. E3.12. Determine the approximate flowrate per unit width of the channel.



■ FIGURE E3.12

SOLUTION

Under the assumptions of steady, inviscid, incompressible flow, we can apply Eq. 3.21 to obtain Q/b , the flowrate per unit width, as

$$\frac{Q}{b} = z_2 \sqrt{\frac{2g(z_1 - z_2)}{1 - (z_2/z_1)^2}}$$

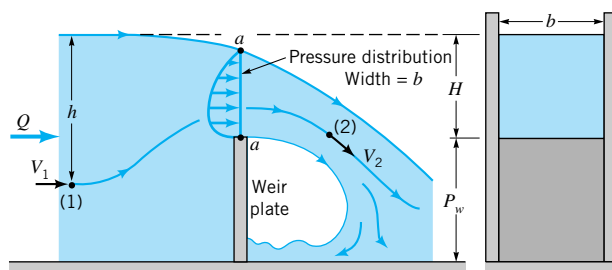
In this instance $z_1 = 5.0$ m and $a = 0.80$ m so the ratio $a/z_1 = 0.16 < 0.20$, and we can assume that the contraction coefficient is approximately $C_c = 0.61$. Thus, $z_2 = C_c a = 0.61(0.80 \text{ m}) = 0.488$ m and we obtain the flowrate

$$\begin{aligned} \frac{Q}{b} &= (0.488 \text{ m}) \sqrt{\frac{2(9.81 \text{ m/s}^2)(5.0 \text{ m} - 0.488 \text{ m})}{1 - (0.488 \text{ m}/5.0 \text{ m})^2}} \\ &= 4.61 \text{ m}^2/\text{s} \end{aligned} \quad (\text{Ans})$$

If we consider $z_1 \gg z_2$ and neglect the kinetic energy of the upstream fluid, we would have

$$\frac{Q}{b} = z_2 \sqrt{2gz_1} = 0.488 \text{ m} \sqrt{2(9.81 \text{ m/s}^2)(5.0 \text{ m})} = 4.83 \text{ m}^2/\text{s}$$

In this case the difference in Q with or without including V_1 is not too significant because the depth ratio is fairly large ($z_1/z_2 = 5.0/0.488 = 10.2$). Thus, it is often reasonable to neglect the kinetic energy upstream from the gate compared to that downstream of it.



■ **FIGURE 3.20** Rectangular, sharp-crested weir geometry.

Another device used to measure flow in an open channel is a *weir*. A typical rectangular, sharp-crested weir is shown in Fig. 3.20. For such devices the flowrate of liquid over the top of the weir plate is dependent on the weir height, P_w , the width of the channel, b , and the head, H , of the water above the top of the weir. Application of the Bernoulli equation can provide a simple approximation of the flowrate expected for these situations, even though the actual flow is quite complex.

Between points (1) and (2) the pressure and gravitational fields cause the fluid to accelerate from velocity V_1 to velocity V_2 . At (1) the pressure is $p_1 = \gamma h$, while at (2) the pressure is essentially atmospheric, $p_2 = 0$. Across the curved streamlines directly above the top of the weir plate (section $a-a$), the pressure changes from atmospheric on the top surface to some maximum value within the fluid stream and then to atmospheric again at the bottom surface. This distribution is indicated in Fig. 3.20. Such a pressure distribution, combined with the streamline curvature and gravity, produces a rather nonuniform velocity profile across this section. This velocity distribution can be obtained from experiments or a more advanced theory.

For now, we will take a very simple approach and assume that the weir flow is similar in many respects to an orifice-type flow with a free streamline. In this instance we would expect the average velocity across the top of the weir to be proportional to $\sqrt{2gH}$ and the flow area for this rectangular weir to be proportional to Hb . Hence, it follows that

$$Q = C_1 H b \sqrt{2gH} = C_1 b \sqrt{2g} H^{3/2}$$

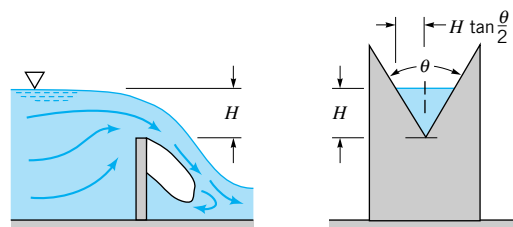
where C_1 is a constant to be determined.

Simple use of the Bernoulli equation has provided a method to analyze the relatively complex flow over a weir. The correct functional dependence of Q on H has been obtained ($Q \sim H^{3/2}$), but the value of the coefficient C_1 is unknown. Even a more advanced analysis cannot predict its value accurately. As is discussed in **Chapter 10**, experiments are used to determine the value of C_1 .

Flowrate over a weir is a function of the head on the weir.

EXAMPLE 3.13

Water flows over a triangular weir, as is shown in Fig. E3.13. Based on a simple analysis using the Bernoulli equation, determine the dependence of the flowrate on the depth H . If the flowrate is Q_0 when $H = H_0$, estimate the flowrate when the depth is increased to $H = 3H_0$.



■ **FIGURE E3.13**

SOLUTION

With the assumption that the flow is steady, inviscid, and incompressible, it is reasonable to assume from Eq. 3.18 that the average speed of the fluid over the triangular notch in the weir plate is proportional to $\sqrt{2gH}$. Also, the flow area for a depth of H is $H [H \tan(\theta/2)]$. The combination of these two ideas gives

$$Q = AV = H^2 \tan \frac{\theta}{2} (C_2 \sqrt{2gH}) = C_2 \tan \frac{\theta}{2} \sqrt{2g} H^{5/2} \quad (\text{Ans})$$

where C_2 is an unknown constant to be determined experimentally.

Thus, an increase in the depth by a factor of three (from H_0 to $3H_0$) results in an increase of the flowrate by a factor of

$$\begin{aligned} \frac{Q_{3H_0}}{Q_{H_0}} &= \frac{C_2 \tan(\theta/2) \sqrt{2g} (3H_0)^{5/2}}{C_2 \tan(\theta/2) \sqrt{2g} (H_0)^{5/2}} \\ &= 15.6 \end{aligned} \quad (\text{Ans})$$

Note that for a triangular weir the flowrate is proportional to $H^{5/2}$, whereas for the rectangular weir discussed above, it is proportional to $H^{3/2}$. The triangular weir can be accurately used over a wide range of flowrates.

3.7 The Energy Line and the Hydraulic Grade Line

The hydraulic grade line and energy line are graphical forms of the Bernoulli equation.

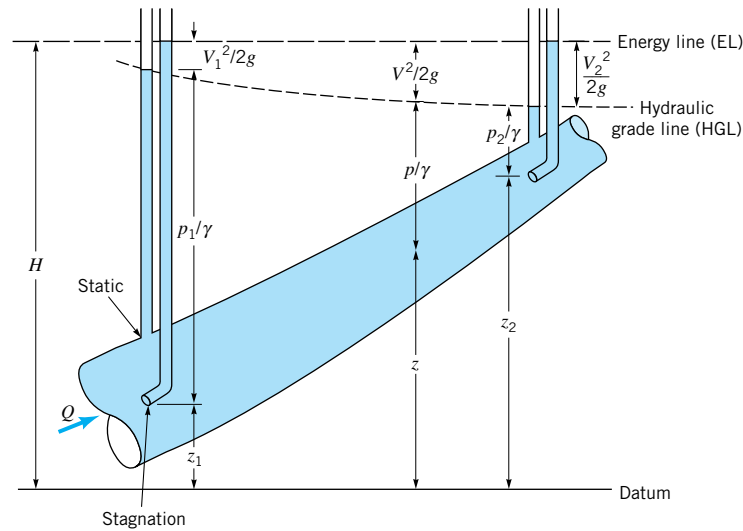
As was discussed in Section 3.4, the Bernoulli equation is actually an energy equation representing the partitioning of energy for an inviscid, incompressible, steady flow. The sum of the various energies of the fluid remains constant as the fluid flows from one section to another. A useful interpretation of the Bernoulli equation can be obtained through the use of the concepts of the *hydraulic grade line* (HGL) and the *energy line* (EL). These ideas represent a geometrical interpretation of a flow and can often be effectively used to better grasp the fundamental processes involved.

For steady, inviscid, incompressible flow the total energy remains constant along a streamline. The concept of “head” was introduced by dividing each term in Eq. 3.7 by the specific weight, $\gamma = \rho g$, to give the Bernoulli equation in the following form

$$\frac{p}{\gamma} + \frac{V^2}{2g} + z = \text{constant on a streamline} = H \quad (3.22)$$

Each of the terms in this equation has the units of length (feet or meters) and represents a certain type of head. The Bernoulli equation states that the sum of the pressure head, the velocity head, and the elevation head is constant along a streamline. This constant is called the *total head*, H .

The energy line is a line that represents the total head available to the fluid. As shown in Fig. 3.21, the elevation of the energy line can be obtained by measuring the stagnation pressure with a Pitot tube. (A Pitot tube is the portion of a Pitot-static tube that measures the stagnation pressure. See [Section 3.5](#).) The stagnation point at the end of the Pitot tube provides a measurement of the total head (or energy) of the flow. The static pressure tap connected to the piezometer tube shown, on the other hand, measures the sum of the pressure head and the elevation head, $p/\gamma + z$. This sum is often called the *piezometric head*. The static pressure tap does not measure the velocity head.



■ **FIGURE 3.21** Representation of the energy line and the hydraulic grade line.

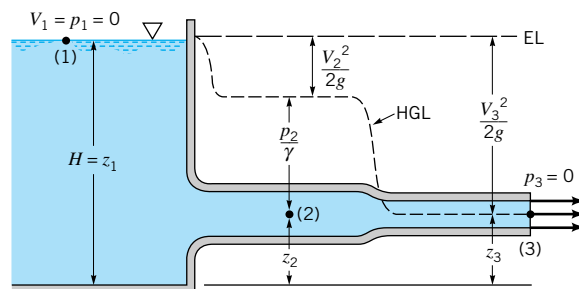
According to Eq. 3.22, the total head remains constant along the streamline (provided the assumptions of the Bernoulli equation are valid). Thus, a Pitot tube at any other location in the flow will measure the same total head, as is shown in the figure. The elevation head, velocity head, and pressure head may vary along the streamline, however.

The locus of elevations provided by a series of Pitot tubes is termed the energy line, EL. The locus provided by a series of piezometer taps is termed the hydraulic grade line, HGL. Under the assumptions of the Bernoulli equation, the energy line is horizontal. If the fluid velocity changes along the streamline, the hydraulic grade line will not be horizontal. If viscous effects are important (as they often are in pipe flows), the total head does not remain constant due to a loss in energy as the fluid flows along its streamline. This means that the energy line is no longer horizontal. Such viscous effects are discussed in Chapter 8.

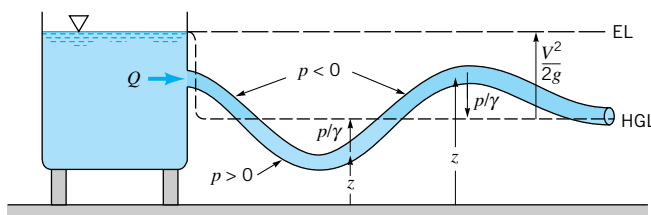
The energy line and hydraulic grade line for flow from a large tank are shown in Fig. 3.22. If the flow is steady, incompressible, and inviscid, the energy line is horizontal and at the elevation of the liquid in the tank (since the fluid velocity in the tank and the pressure on the surface are zero). The hydraulic grade line lies a distance of one velocity head, $V^2/2g$, below the energy line. Thus, a change in fluid velocity due to a change in the pipe diameter results in a change in the elevation of the hydraulic grade line. At the pipe outlet the pressure head is zero (gage) so the pipe elevation and the hydraulic grade line coincide.

The distance from the pipe to the hydraulic grade line indicates the pressure within the pipe, as is shown in Fig. 3.23. If the pipe lies below the hydraulic grade line, the pressure

The hydraulic grade line lies one velocity head below the energy line.



■ **FIGURE 3.22** The energy line and hydraulic grade line for flow from a tank.



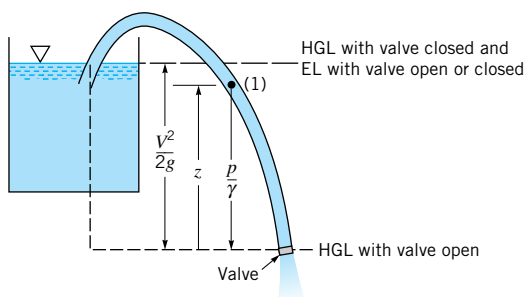
■ **FIGURE 3.23** Use of the energy line and the hydraulic grade line.

For flow below (above) the hydraulic grade line, the pressure is positive (negative).

within the pipe is positive (above atmospheric). If the pipe lies above the hydraulic grade line, the pressure is negative (below atmospheric). Thus, a scale drawing of a pipeline and the hydraulic grade line can be used to readily indicate regions of positive or negative pressure within a pipe.

EXAMPLE 3.14

Water is siphoned from the tank shown in Fig. E3.14 through a hose of constant diameter. A small hole is found in the hose at location (1) as indicated. When the siphon is used, will water leak out of the hose, or will air leak into the hose, thereby possibly causing the siphon to malfunction?



■ **FIGURE E3.14**

SOLUTION

Whether air will leak into or water will leak out of the hose depends on whether the pressure within the hose at (1) is less than or greater than atmospheric. Which happens can be easily determined by using the energy line and hydraulic grade line concepts as follows. With the assumption of steady, incompressible, inviscid flow it follows that the total head is constant—thus, the energy line is horizontal.

Since the hose diameter is constant, it follows from the continuity equation ($AV = \text{constant}$) that the water velocity in the hose is constant throughout. Thus, the hydraulic grade line is a constant distance, $V^2/2g$, below the energy line as shown in Fig. E3.14. Since the pressure at the end of the hose is atmospheric, it follows that the hydraulic grade line is at the same elevation as the end of the hose outlet. The fluid within the hose at any point above the hydraulic grade line will be at less than atmospheric pressure.

Thus, air will leak into the hose through the hole at point (1).

(Ans)

In practice, viscous effects may be quite important, making this simple analysis (horizontal energy line) incorrect. However, if the hose is “not too small diameter,” “not too long,” the fluid “not too viscous,” and the flowrate “not too large,” the above result may be very accurate. If any of these assumptions are relaxed, a more detailed analysis is required (see

Chapter 8). If the end of the hose were closed so that the flowrate were zero, the hydraulic grade line would coincide with the energy line ($V^2/2g = 0$ throughout), the pressure at (1) would be greater than atmospheric, and water would leak through the hole at (1).

The above discussion of the hydraulic grade line and the energy line is restricted to ideal situations involving inviscid, incompressible flows. Another restriction is that there are no “sources” or “sinks” of energy within the flow field. That is, there are no pumps or turbines involved. Alterations in the energy line and hydraulic grade line concepts due to these devices are discussed in Chapter 8.

3.8 Restrictions on Use of the Bernoulli Equation

Proper use of the Bernoulli equation requires close attention to the assumptions used in its derivation. In this section we review some of these assumptions and consider the consequences of incorrect use of the equation.

3.8.1 Compressibility Effects

One of the main assumptions is that the fluid is incompressible. Although this is reasonable for most liquid flows, it can, in certain instances, introduce considerable errors for gases.

In the previous section, we saw that the stagnation pressure is greater than the static pressure by an amount $\rho V^2/2$, provided that the density remains constant. If this dynamic pressure is not too large compared with the static pressure, the density change between two points is not very large and the flow can be considered incompressible. However, since the dynamic pressure varies as V^2 , the error associated with the assumption that a fluid is incompressible increases with the square of the velocity of the fluid. To account for compressibility effects we must return to Eq. 3.6 and properly integrate the term $\int dp/\rho$ when ρ is not constant.

The Bernoulli equation can be modified for compressible flows.

A simple, although specialized, case of compressible flow occurs when the temperature of a perfect gas remains constant along the streamline—*isothermal flow*. Thus, we consider $p = \rho RT$, where T is constant. (In general, p , ρ , and T will vary.) For steady, inviscid, isothermal flow, Eq. 3.6 becomes

$$RT \int \frac{dp}{p} + \frac{1}{2} V^2 + gz = \text{constant}$$

where we have used $\rho = p/RT$. The pressure term is easily integrated and the constant of integration evaluated if z_1 , p_1 , and V_1 are known at some location on the streamline. The result is

$$\frac{V_1^2}{2g} + z_1 + \frac{RT}{g} \ln \left(\frac{p_1}{p_2} \right) = \frac{V_2^2}{2g} + z_2 \quad (3.23)$$

Equation 3.23 is the inviscid, isothermal analog of the incompressible Bernoulli equation. In the limit of small pressure difference, $p_1/p_2 = 1 + (p_1 - p_2)/p_2 = 1 + \varepsilon$, with $\varepsilon \ll 1$ and Eq. 3.23 reduces to the standard incompressible Bernoulli equation. This can be shown by use of the approximation $\ln(1 + \varepsilon) \approx \varepsilon$ for small ε . The use of Eq. 3.23 in practical applications is restricted by the inviscid flow assumption, since (as is discussed in Section 11.5) most isothermal flows are accompanied by viscous effects.

A much more common compressible flow condition is that of *isentropic* (constant entropy) flow of a perfect gas. Such flows are reversible adiabatic processes—“no friction or

heat transfer”—and are closely approximated in many physical situations. As discussed fully in **Chapter 11**, for isentropic flow of a perfect gas the density and pressure are related by $p/\rho^k = C$, where k is the specific heat ratio and C is a constant. Hence, the $\int dp/\rho$ integral of Eq. 3.6 can be evaluated as follows. The density can be written in terms of the pressure as $\rho = p^{1/k}C^{-1/k}$ so that Eq. 3.6 becomes

$$C^{1/k} \int p^{-1/k} dp + \frac{1}{2} V^2 + gz = \text{constant}$$

The pressure term can be integrated between points (1) and (2) on the streamline and the constant C evaluated at either point ($C^{1/k} = p_1^{1/k}/\rho_1$ or $C^{1/k} = p_2^{1/k}/\rho_2$) to give the following:

$$\begin{aligned} C^{1/k} \int_{p_1}^{p_2} p^{-1/k} dp &= C^{1/k} \left(\frac{k}{k-1} \right) [p_2^{(k-1)/k} - p_1^{(k-1)/k}] \\ &= \left(\frac{k}{k-1} \right) \left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} \right) \end{aligned}$$

Thus, the final form of Eq. 3.6 for compressible, isentropic, steady flow of a perfect gas is

$$\left(\frac{k}{k-1} \right) \frac{p_1}{\rho_1} + \frac{V_1^2}{2} + gz_1 = \left(\frac{k}{k-1} \right) \frac{p_2}{\rho_2} + \frac{V_2^2}{2} + gz_2 \quad (3.24)$$

An equation similar to the Bernoulli equation can be obtained for isentropic flow of a perfect gas.

The similarities between the results for compressible isentropic flow (Eq. 3.24) and incompressible isentropic flow (the Bernoulli equation, Eq. 3.7) are apparent. The only differences are the factors of $[k/(k-1)]$ that multiply the pressure terms and the fact that the densities are different ($\rho_1 \neq \rho_2$). In the limit of “low-speed flow” the two results are exactly the same as is seen by the following.

We consider the stagnation point flow of Section 3.5 to illustrate the difference between the incompressible and compressible results. As is shown in Chapter 11, Eq. 3.24 can be written in dimensionless form as

$$\frac{p_2 - p_1}{p_1} = \left[\left(1 + \frac{k-1}{2} \text{Ma}_1^2 \right)^{k/(k-1)} - 1 \right] \quad (\text{compressible}) \quad (3.25)$$

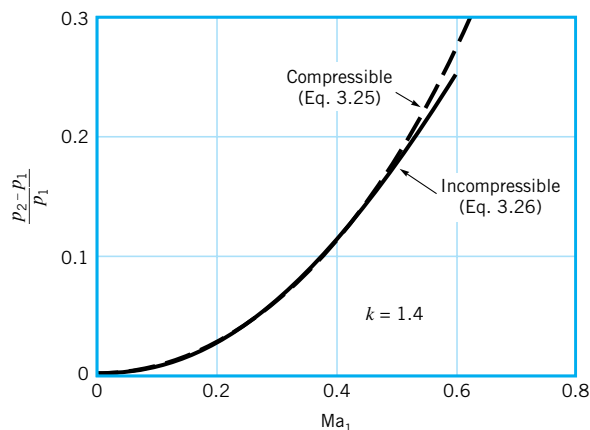
where (1) denotes the upstream conditions and (2) the stagnation conditions. We have assumed $z_1 = z_2$, $V_2 = 0$, and have denoted $\text{Ma}_1 = V_1/c_1$ as the upstream *Mach number*—the ratio of the fluid velocity to the speed of sound, $c_1 = \sqrt{kRT_1}$.

A comparison between this compressible result and the incompressible result is perhaps most easily seen if we write the incompressible flow result in terms of the pressure ratio and the Mach number. Thus, we divide each term in the Bernoulli equation, $\rho V_1^2/2 + p_1 = p_2$, by p_1 and use the perfect gas law, $p_1 = \rho RT_1$, to obtain

$$\frac{p_2}{p_1} = \frac{V_1^2}{2RT_1}$$

Since $\text{Ma}_1 = V_1/\sqrt{kRT_1}$ this can be written as

$$\frac{p_2 - p_1}{p_1} = \frac{k\text{Ma}_1^2}{2} \quad (\text{incompressible}) \quad (3.26)$$



■ **FIGURE 3.24** Pressure ratio as a function of Mach number for incompressible and compressible (isentropic) flow.

Equations 3.25 and 3.26 are plotted in Fig. 3.24. In the low-speed limit of $Ma_1 \rightarrow 0$, both of the results are the same. This can be seen by denoting $(k - 1)Ma_1^2/2 = \tilde{\epsilon}$ and using the binomial expansion, $(1 + \tilde{\epsilon})^n = 1 + n\tilde{\epsilon} + n(n - 1)\tilde{\epsilon}^2/2 + \cdots$, where $n = k/(k - 1)$, to write Eq. 3.25 as

$$\frac{p_2 - p_1}{p_1} = \frac{kMa_1^2}{2} \left(1 + \frac{1}{4}Ma_1^2 + \frac{2 - k}{24}Ma_1^4 + \cdots \right) \text{ (compressible)}$$

For small Mach numbers the compressible and incompressible results are nearly the same.

For $Ma_1 \ll 1$ this compressible flow result agrees with Eq. 3.26. The incompressible and compressible equations agree to within about 2% up to a Mach number of approximately $Ma_1 = 0.3$. For larger Mach numbers the disagreement between the two results increases.

Thus, a “rule of thumb” is that the flow of a perfect gas may be considered as incompressible provided the Mach number is less than about 0.3. In standard air ($T_1 = 59^\circ\text{F}$, $c_1 = \sqrt{kRT_1} = 1117 \text{ ft/s}$) this corresponds to a speed of $V_1 = c_1 Ma_1 = 0.3(1117 \text{ ft/s}) = 335 \text{ ft/s} = 228 \text{ mi/hr}$. At higher speeds, compressibility may become important.

EXAMPLE 3.15

A Boeing 777 flies at Mach 0.82 at an altitude of 10 km in a standard atmosphere. Determine the stagnation pressure on the leading edge of its wing if the flow is incompressible; and if the flow is compressible isentropic.

SOLUTION

From Tables 1.8 and C.2 we find that $p_1 = 26.5 \text{ kPa (abs)}$, $T_1 = -49.9^\circ\text{C}$, $\rho = 0.414 \text{ kg/m}^3$, and $k = 1.4$. Thus, if we assume incompressible flow, Eq. 3.26 gives

$$\frac{p_2 - p_1}{p_1} = \frac{kMa_1^2}{2} = 1.4 \frac{(0.82)^2}{2} = 0.471$$

or

$$p_2 - p_1 = 0.471 (26.5 \text{ kPa}) = 12.5 \text{ kPa} \quad (\text{Ans})$$

On the other hand, if we assume isentropic flow, Eq. 3.25 gives

$$\frac{p_2 - p_1}{p_1} = \left\{ \left[1 + \frac{(1.4 - 1)}{2} (0.82)^2 \right]^{1.4/(1.4 - 1)} - 1 \right\} = 0.555$$

or

$$p_2 - p_1 = 0.555 (26.5 \text{ kPa}) = 14.7 \text{ kPa} \quad (\text{Ans})$$

We see that at Mach 0.82 compressibility effects are of importance. The pressure (and, to a first approximation, the lift and drag on the airplane; see Chapter 9) is approximately $14.7/12.5 = 1.18$ times greater according to the compressible flow calculations. This may be very significant. As discussed in **Chapter 11**, for Mach numbers greater than 1 (supersonic flow) the differences between incompressible and compressible results are often not only quantitative but also qualitative.

Note that if the airplane were flying at Mach 0.30 (rather than 0.82) the corresponding values would be $p_2 - p_1 = 1.670 \text{ kPa}$ for incompressible flow and $p_2 - p_1 = 1.707 \text{ kPa}$ for compressible flow. The difference between these two results is about 2 percent.

3.8.2 Unsteady Effects

Another restriction of the Bernoulli equation (Eq. 3.7) is the assumption that the flow is steady. For such flows, on a given streamline the velocity is a function of only s , the location along the streamline. That is, along a streamline $V = V(s)$. For unsteady flows the velocity is also a function of time, so that along a streamline $V = V(s, t)$. Thus when taking the time derivative of the velocity to obtain the streamwise acceleration, we obtain $a_s = \partial V / \partial t + V \partial V / \partial s$ rather than just $a_s = V \partial V / \partial s$ as is true for steady flow. For steady flows the acceleration is due to the change in velocity resulting from a change in position of the particle (the $V \partial V / \partial s$ term), whereas for unsteady flow there is an additional contribution to the acceleration resulting from a change in velocity with time at a fixed location (the $\partial V / \partial t$ term). These effects are discussed in detail in **Chapter 4**. The net effect is that the inclusion of the unsteady term, $\partial V / \partial t$, does not allow the equation of motion to be easily integrated (as was done to obtain the Bernoulli equation) unless additional assumptions are made.

The Bernoulli equation was obtained by integrating the component of Newton's second law (Eq. 3.5) along the streamline. When integrated, the acceleration contribution to this equation, the $\frac{1}{2} \rho d(V^2)$ term, gave rise to the kinetic energy term in the Bernoulli equation. If the steps leading to Eq. 3.5 are repeated with the inclusion of the unsteady effect ($\partial V / \partial t \neq 0$) the following is obtained:

$$\rho \frac{\partial V}{\partial t} ds + dp + \frac{1}{2} \rho d(V^2) + \gamma dz = 0 \quad (\text{along a streamline})$$

For incompressible flow this can be easily integrated between points (1) and (2) to give

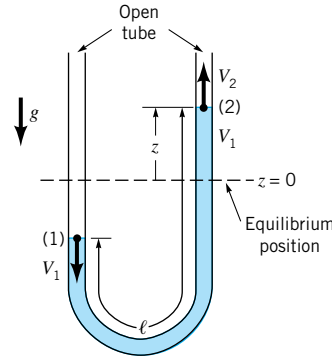
$$p_1 + \frac{1}{2} \rho V_1^2 + \gamma z_1 = \rho \int_{s_1}^{s_2} \frac{\partial V}{\partial t} ds + p_2 + \frac{1}{2} \rho V_2^2 + \gamma z_2 \quad (\text{along a streamline}) \quad (3.27)$$

Equation 3.27 is an unsteady form of the Bernoulli equation valid for unsteady, incompressible, inviscid flow. Except for the integral involving the local acceleration, $\partial V / \partial t$, it is identical to the steady Bernoulli equation. In general, it is not easy to evaluate this integral because the variation of $\partial V / \partial t$ along the streamline is not known. In some situations the concepts of “irrotational flow” and the “velocity potential” can be used to simplify this integral. These topics are discussed in Chapter 6.

The Bernoulli equation can be modified for unsteady flows.

EXAMPLE 3.16

An incompressible, inviscid liquid is placed in a vertical, constant-diameter U-tube as indicated in Fig. E3.16. When released from the nonequilibrium position shown, the liquid column will oscillate at a specific frequency. Determine this frequency.



■ FIGURE E3.16

SOLUTION

The frequency of oscillation can be calculated by use of Eq. 3.27 as follows. Let points (1) and (2) be at the air–water interfaces of the two columns of the tube and $z = 0$ correspond to the equilibrium position of these interfaces. Hence, $p_1 = p_2 = 0$ and if $z_2 = z$, then $z_1 = -z$. In general, z is a function of time, $z = z(t)$. For a constant diameter tube, at any instant in time the fluid speed is constant throughout the tube, $V_1 = V_2 = V$, and the integral representing the unsteady effect in Eq. 3.27 can be written as

$$\int_{s_1}^{s_2} \frac{\partial V}{\partial t} ds = \frac{dV}{dt} \int_{s_1}^{s_2} ds = \ell \frac{dV}{dt}$$

where ℓ is the total length of the liquid column as shown in the figure. Thus, Eq. 3.27 can be written as

$$\gamma(-z) = \rho \ell \frac{dV}{dt} + \gamma z$$

Since $V = dz/dt$ and $\gamma = \rho g$, this can be written as the second-order differential equation describing simple harmonic motion

$$\frac{d^2 z}{dt^2} + \frac{2g}{\ell} z = 0$$

which has the solution $z(t) = C_1 \sin(\sqrt{2g/\ell} t) + C_2 \cos(\sqrt{2g/\ell} t)$. The values of the constants C_1 and C_2 depend on the initial state (velocity and position) of the liquid at $t = 0$. Thus, the liquid oscillates in the tube with a frequency

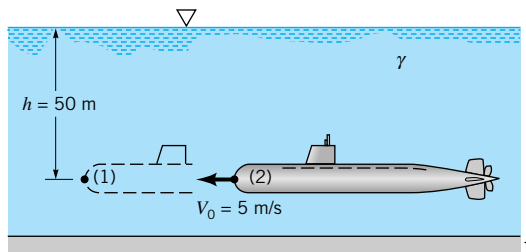
$$\omega = \sqrt{2g/\ell} \quad (\text{Ans})$$

This frequency depends on the length of the column and the acceleration of gravity (in a manner very similar to the oscillation of a pendulum). The period of this oscillation (the time required to complete an oscillation) is $t_0 = 2\pi\sqrt{\ell/2g}$.

In a few unsteady flow cases, the flow can be made steady by an appropriate selection of the coordinate system. Example 3.17 illustrates this.

EXAMPLE 3.17

A submarine moves through the seawater ($SG = 1.03$) at a depth of 50 m with velocity $V_0 = 5.0$ m/s as shown in Fig. E3.17. Determine the pressure at the stagnation point (2).



■ FIGURE E3.17

SOLUTION

In a coordinate system fixed to the ground, the flow is unsteady. For example, the water velocity at (1) is zero with the submarine in its initial position, but at the instant when the nose, (2), reaches point (1) the velocity there becomes $\mathbf{V}_1 = -V_0\hat{\mathbf{i}}$. Thus, $\partial\mathbf{V}_1/\partial t \neq 0$ and the flow is unsteady. Application of the steady Bernoulli equation between (1) and (2) would give the incorrect result that “ $p_1 = p_2 + \rho V_0^2/2$.” According to this result the static pressure is greater than the stagnation pressure—an incorrect use of the Bernoulli equation.

We can either use an unsteady analysis for the flow (which is outside the scope of this text) or redefine the coordinate system so that it is fixed on the submarine, giving steady flow with respect to this system. The correct method would be

$$\begin{aligned} p_2 &= \frac{\rho V_1^2}{2} + \gamma h = [(1.03)(1000) \text{ kg/m}^3] (5.0 \text{ m/s})^2/2 \\ &\quad + (9.80 \times 10^3 \text{ N/m}^3)(1.03)(50 \text{ m}) \\ &= (12,900 + 505,000) \text{ N/m}^2 = 518 \text{ kPa} \end{aligned} \quad (\text{Ans})$$

similar to that discussed in Example 3.2.

If the submarine were accelerating, $\partial V_0/\partial t \neq 0$, the flow would be unsteady in either of the above coordinate systems and we would be forced to use an unsteady form of the Bernoulli equation.

Some unsteady flows may be treated as “quasisteady” and solved approximately by using the steady Bernoulli equation. In these cases the unsteadiness is “not too great” (in some sense), and the steady flow results can be applied at each instant in time as though the flow were steady. The slow draining of a tank filled with liquid provides an example of this type of flow.

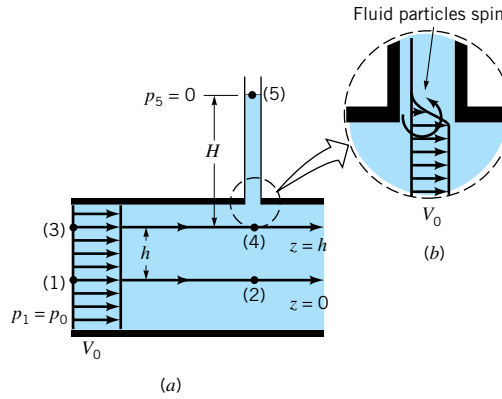
3.8.3 Rotational Effects

Another of the restrictions of the Bernoulli equation is that it is applicable along the streamline. Application of the Bernoulli equation across streamlines (i.e., from a point on one streamline to a point on another streamline) can lead to considerable errors, depending on the particular flow conditions involved. In general, the Bernoulli constant varies from streamline to streamline. However, under certain restrictions this constant is the same throughout the entire flow field. Example 3.18 illustrates this fact.

Care must be used in applying the Bernoulli equation across streamlines.

EXAMPLE 3.18

Consider the uniform flow in the channel shown in Fig. E3.18a. Discuss the use of the Bernoulli equation between points (1) and (2), points (3) and (4), and points (4) and (5). The liquid in the vertical piezometer tube is stationary.



■ FIGURE E3.18

SOLUTION

If the flow is steady, inviscid, and incompressible, Eq. 3.7 written between points (1) and (2) gives

$$p_1 + \frac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \gamma z_2 = \text{constant} = C_{12}$$

Since $V_1 = V_2 = V_0$ and $z_1 = z_2 = 0$, it follows that $p_1 = p_2 = p_0$ and the Bernoulli constant for this streamline, C_{12} , is given by

$$C_{12} = \frac{1}{2}\rho V_0^2 + p_0$$

Along the streamline from (3) to (4) we note that $V_3 = V_4 = V_0$ and $z_3 = z_4 = h$. As was shown in Example 3.5, application of $\mathbf{F} = m\mathbf{a}$ across the streamline (Eq. 3.12) gives $p_3 = p_1 - \gamma h$ because the streamlines are straight and horizontal. The above facts combined with the Bernoulli equation applied between (3) and (4) show that $p_3 = p_4$ and that the Bernoulli constant along this streamline is the same as that along the streamline between (1) and (2). That is, $C_{34} = C_{12}$, or

$$p_3 + \frac{1}{2}\rho V_3^2 + \gamma z_3 = p_4 + \frac{1}{2}\rho V_4^2 + \gamma z_4 = C_{34} = C_{12}$$

Similar reasoning shows that the Bernoulli constant is the same for any streamline in Fig. E3.18. Hence,

$$p + \frac{1}{2}\rho V^2 + \gamma z = \text{constant throughout the flow in the channel}$$

Again from Example 3.5 we recall that

$$p_4 = p_5 + \gamma H = \gamma H$$

If we apply the Bernoulli equation across streamlines from (4) to (5), we obtain the incorrect result “ $H = p_4/\gamma + V_4^2/2g$.” The correct result is $H = p_4/\gamma$.

From the above we see that we can apply the Bernoulli equation across streamlines (1)–(2) and (3)–(4) (that is, $C_{12} = C_{34}$) but not across streamlines from (4) to (5). The reason for this is that while the flow in the channel is “irrotational,” it is “rotational” between the flowing fluid in the channel and the stationary fluid in the piezometer tube. Because of the uniform velocity profile across the channel, it is seen that the fluid particles do not rotate or “spin” as they move. The flow is “irrotational.” However, as seen in Fig. E3.18b, there is a

very thin shear layer between (4) and (5) in which adjacent fluid particles interact and rotate or “spin.” This produces a “rotational” flow. A more complete analysis would show that the Bernoulli equation cannot be applied across streamlines if the flow is “rotational” (see Chapter 6).

As is suggested by Example 3.18, if the flow is “irrotational” (that is, the fluid particles do not “spin” as they move), it is appropriate to use the Bernoulli equation across streamlines. However, if the flow is “rotational” (fluid particles “spin”), use of the Bernoulli equation is restricted to flow along a streamline. The distinction between irrotational and rotational flow is often a very subtle and confusing one. These topics are discussed in more detail in Chapter 6. A thorough discussion can be found in more advanced texts (Ref. 3).

3.8.4 Other Restrictions

Another restriction on the Bernoulli equation is that the flow is inviscid. As is discussed in Section 3.4, the Bernoulli equation is actually a first integral of Newton’s second law along a streamline. This general integration was possible because, in the absence of viscous effects, the fluid system considered was a conservative system. The total energy of the system remains constant. If viscous effects are important the system is nonconservative and energy losses occur. A more detailed analysis is needed for these cases. Such material is presented in Chapter 5.

The Bernoulli equation is not valid for flows that involve pumps or turbines.

The final basic restriction on use of the Bernoulli equation is that there are no mechanical devices (pumps or turbines) in the system between the two points along the streamline for which the equation is applied. These devices represent sources or sinks of energy. Since the Bernoulli equation is actually one form of the energy equation, it must be altered to include pumps or turbines, if these are present. The inclusion of pumps and turbines is covered in Chapters 5 and 12.

In this chapter we have spent considerable time investigating fluid dynamic situations governed by a relatively simple analysis for steady, inviscid, incompressible flows. Many flows can be adequately analyzed by use of these ideas. However, because of the rather severe restrictions imposed, many others cannot. An understanding of these basic ideas will provide a firm foundation for the remainder of the topics in this book.

Key Words and Topics

In the E-book, click on any key word or topic to go to that subject.

Bernoulli equation
Cavitation
Continuity equation
Energy line
 $F = ma$ normal to streamline
 $F = ma$ along streamline

Flowrate measurement
Free jet
Head
Hydraulic grade line
Mass flowrate
Newton’s second law
Nozzle meter
Orifice meter

Pilot tube
Stagnation pressure
Static pressure
Streamline
Venturi meter
Volume flowrate
Weir

References

1. Riley, W. F., and Sturges, L. D., *Engineering Mechanics: Dynamics*, 2nd Ed., Wiley, New York, 1996.
2. Tipler, P. A., *Physics*, Worth, New York, 1982.
3. Panton, R. L., *Incompressible Flow*, Wiley, New York, 1984.

Review Problems

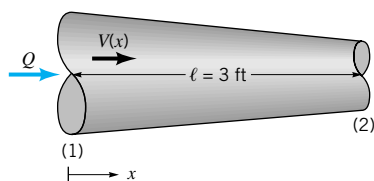
In the E-book, [click here](#) to go to a set of review problems complete with answers and detailed solutions.

Problems

Note: Unless otherwise indicated use the values of fluid properties found in the tables on the inside of the front cover. Problems designated with an (*) are intended to be solved with the aid of a programmable calculator or a computer. Problems designated with a (†) are “open-ended” problems and require critical thinking in that to work them one must make various assumptions and provide the necessary data. There is not a unique answer to these problems.

In the E-book, answers to the even-numbered problems can be obtained by clicking on the problem number. In the E-book, access to the videos that accompany problems can be obtained by clicking on the “video” segment (i.e., **Video 3.3**) of the problem statement. The lab-type problems can be accessed by clicking on the “[click here](#)” segment of the problem statement.

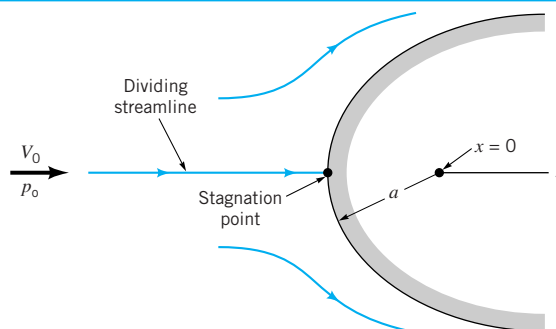
3.1 Water flows steadily through the variable area horizontal pipe shown in Fig. P3.1. The centerline velocity is given by $\mathbf{V} = 10(1 + x)\hat{\mathbf{i}}$ ft/s, where x is in feet. Viscous effects are neglected. **(a)** Determine the pressure gradient, $\partial p/\partial x$, (as a function of x) needed to produce this flow. **(b)** If the pressure at section (1) is 50 psi, determine the pressure at (2) by (i) integration of the pressure gradient obtained in **(a)**, (ii) application of the Bernoulli equation.



■ FIGURE P3.1

3.2 Repeat Problem 3.1 if the pipe is vertical with the flow down.

3.3 An incompressible fluid with density ρ flows steadily past the object shown in **Video V3.3** and Fig. P3.3. The fluid velocity along the horizontal dividing streamline ($-\infty \leq x \leq -a$) is found to be $V = V_0(1 + a/x)$, where a is the radius of curvature of the front of the object and V_0 is the upstream velocity. **(a)** Determine the pressure gradient along this streamline. **(b)** If the upstream pressure is p_0 , integrate the pressure gradient to obtain the pressure $p(x)$ for $-\infty \leq x \leq -a$. **(c)** Show from the result of part **(b)** that the pressure at the stagnation point ($x = -a$) is $p_0 + \rho V_0^2/2$, as expected from the Bernoulli equation.



■ FIGURE P3.3

3.4 What pressure gradient along the streamline, dp/ds , is required to accelerate water in a horizontal pipe at a rate of 30 m/s^2 ?

3.5 At a given location the air speed is 20 m/s and the pressure gradient along the streamline is 100 N/m^3 . Estimate the air speed at a point 0.5 m farther along the streamline.

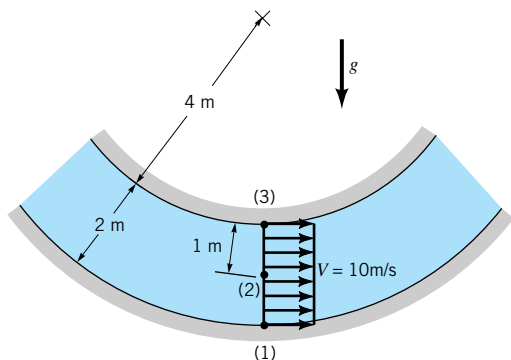
3.6 What pressure gradient along the streamline, dp/ds , is required to accelerate water upward in a vertical pipe at a rate of 30 ft/s^2 ? What is the answer if the flow is downward?

3.7 Consider a compressible fluid for which the pressure and density are related by $p/\rho^n = C_0$, where n and C_0 are constants. Integrate the equation of motion along the streamline, Eq. 3.6, to obtain the “Bernoulli equation” for this compressible flow as $[n/(n-1)]p/\rho + V^2/2 + gz = \text{constant}$.

3.8 The Bernoulli equation is valid for steady, inviscid, incompressible flows with constant acceleration of gravity. Consider flow on a planet where the acceleration of gravity varies with height so that $g = g_0 - cz$, where g_0 and c are constants. Integrate “ $\mathbf{F} = m\mathbf{a}$ ” along a streamline to obtain the equivalent of the Bernoulli equation for this flow.

3.9 Consider a compressible liquid that has a constant bulk modulus. Integrate “ $\mathbf{F} = m\mathbf{a}$ ” along a streamline to obtain the equivalent of the Bernoulli equation for this flow. Assume steady, inviscid flow.

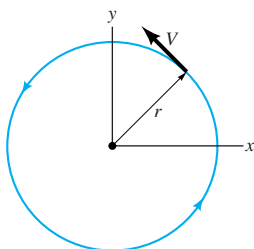
3.10 Water flows around the vertical two-dimensional bend with circular streamlines and constant velocity as shown in Fig. P3.10. If the pressure is 40 kPa at point (1), determine the pressures at points (2) and (3). Assume that the velocity profile is uniform as indicated.



■ FIGURE P3.10

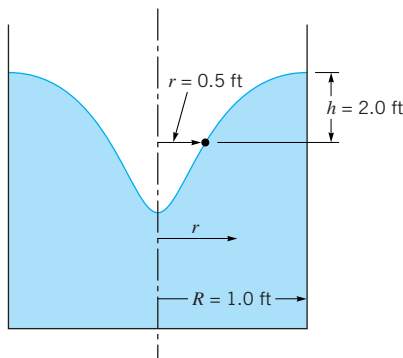
† **3.11** Air flows smoothly past your face as you ride your bike, but bugs and particles of dust pelt your face and get into your eyes. Explain why this is so.

***3.12** Water in a container and air in a tornado flow in horizontal circular streamlines of radius r and speed V as shown in [Video V3.2](#) and Fig. P3.12. Determine the radial pressure gradient, $\partial p / \partial r$, needed for the following situations: **(a)** The fluid is water with $r = 3$ in. and $V = 0.8$ ft/s. **(b)** The fluid is air with $r = 300$ ft and $V = 200$ mph.



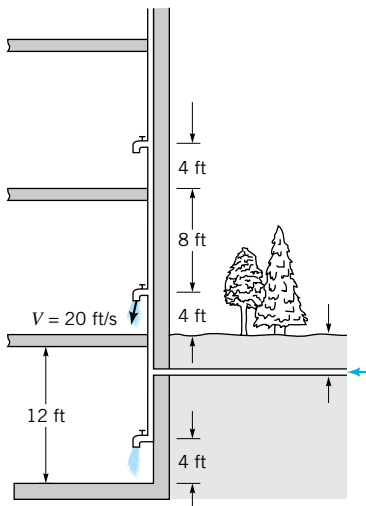
■ FIGURE P3.12

3.13 As shown in Fig. P3.13 and [Video V3.2](#), the swirling motion of a liquid can cause a depression in the free surface. Assume that an inviscid liquid in a tank with an $R = 1.0$ ft radius is rotated sufficiently to produce a free surface that is $h = 2.0$ ft below the liquid at the edge of the tank at a position $r = 0.5$ ft from the center of the tank. Also assume that the liquid velocity is given by $V = K/r$, where K is a constant. **(a)** Show that $h = K^2 [(1/r^2) - (1/R^2)] / (2g)$. **(b)** Determine the value of K for this problem.



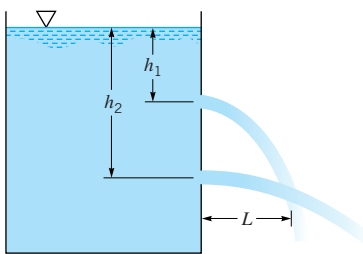
■ FIGURE P3.13

3.14 Water flows from the faucet on the first floor of the building shown in Fig. P3.14 with a maximum velocity of 20 ft/s. For steady inviscid flow, determine the maximum water velocity from the basement faucet and from the faucet on the second floor (assume each floor is 12 ft tall).



■ FIGURE P3.14

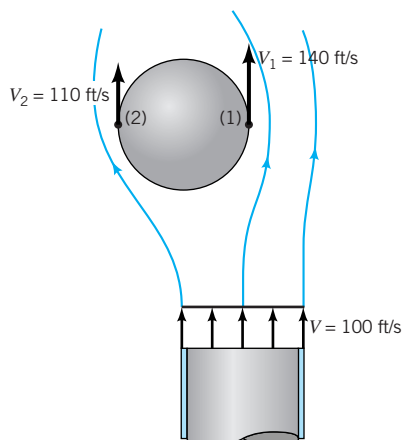
3.15 Water flows from a pop bottle that has holes in it as shown in [Video V3.5](#) and Fig. P3.15. Two streams coming from holes located distances h_1 and h_2 below the free surface intersect at a distance L from the side of the bottle. If viscous effects are negligible and the flow is quasi-steady, show that $L = 2(h_1 h_2)^{1/2}$. Compare this result with experimental data measured from the paused video for which the holes are 2 inches apart.



■ FIGURE P3.15

3.16 A 100 ft/s jet of air flows past a ball as shown in [Video V3.1](#) and Fig. P3.16. When the ball is not centered in the jet, the air velocity is greater on the side of the ball near the jet center [point (1)] than it is on the other side of the ball [point (2)]. Determine the pressure difference, $p_2 - p_1$, across the ball if $V_1 = 140$ ft/s and $V_2 = 110$ ft/s. Neglect gravity and viscous effects.

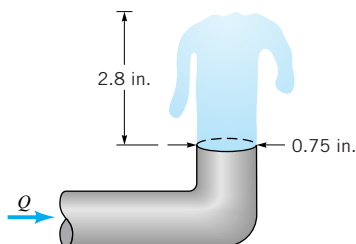
† **3.17** Estimate the pressure needed at the pumper truck in order to shoot water from the street level onto a fire on the roof of a five-story building. List all assumptions and show all calculations.



■ FIGURE P3.16

3.18 A fire hose nozzle has a diameter of $1\frac{1}{8}$ in. According to some fire codes, the nozzle must be capable of delivering at least 250 gal/min. If the nozzle is attached to a 3-in.-diameter hose, what pressure must be maintained just upstream of the nozzle to deliver this flowrate?

3.19 Water flowing from the 0.75-in.-diameter outlet shown in Video V8.6 and Fig. P3.19 rises 2.8 inches above the outlet. Determine the flowrate.

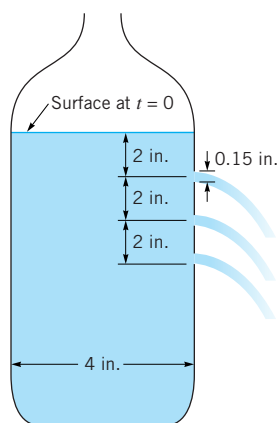


■ FIGURE P3.19

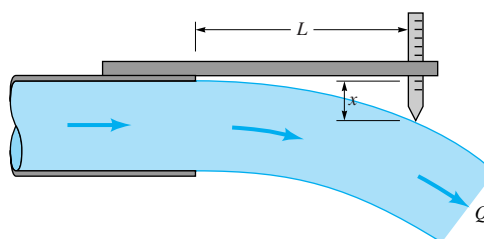
3.20 Pop (with the same properties as water) flows from a 4-in. diameter pop container that contains three holes as shown in Fig. P3.20 (see Video 3.5). The diameter of each fluid stream is 0.15 in., and the distance between holes is 2 in. If viscous effects are negligible and quasi-steady conditions are assumed, determine the time at which the pop stops draining from the top hole. Assume the pop surface is 2 in. above the top hole when $t = 0$. Compare your results with the time you measure from the video.

***3.21** Water flowing from a pipe or a tank is acted upon by gravity and follows a curved trajectory as shown in Fig. P3.21 and Videos V3.5 and V4.3. A simple flow meter can be constructed as shown in Fig. P3.21. A point gage mounted a distance L from the end of the horizontal pipe is adjusted to indicate that the top of the water stream is a distance x below the outlet of the pipe. Show that the flowrate from this pipe of diameter D is given by $Q = \pi D^2 L g^{1/2} / (2^{5/2} x^{1/2})$.

3.22 A person holds her hand out of an open car window while the car drives through still air at 65 mph. Under standard atmospheric conditions, what is the maximum pressure on her



■ FIGURE P3.20



■ FIGURE P3.21

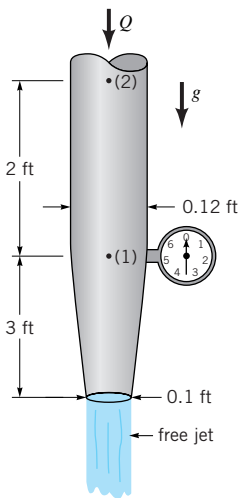
hand? What would be the maximum pressure if the “car” were an Indy 500 racer traveling 220 mph?

3.23 A differential pressure gage attached to a Pitot-static tube (see Video V3.4) is calibrated to give speed rather than the difference between the stagnation and static pressures. The calibration is done so that the speed indicated on the gage is the actual fluid speed if the fluid flowing past the Pitot-static tube is air at standard sea level conditions. Assume the same device is used in water and the gage indicates a speed of 200 knots. Determine the water speed.

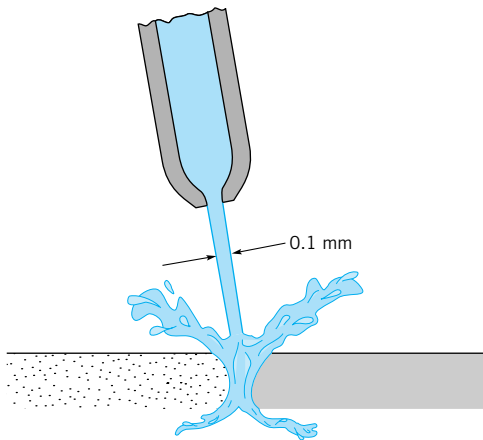
3.24 A 40-mph wind blowing past your house speeds up as it flows up and over the roof. If elevation effects are negligible, determine (a) the pressure at the point on the roof where the speed is 60 mph if the pressure in the free stream blowing toward your house is 14.7 psia. Would this effect tend to push the roof down against the house, or would it tend to lift the roof? (b) Determine the pressure on a window facing the wind if the window is assumed to be a stagnation point.

3.25 Water flows steadily downward through the pipe shown in Fig. P3.25. Viscous effects are negligible, and the pressure gage indicates the pressure is zero at point (1). Determine the flowrate and the pressure at point (2).

3.26 Small-diameter, high-pressure liquid jets can be used to cut various materials as shown in Fig. P3.26. If viscous effects are negligible, estimate the pressure needed to produce a 0.10-mm-diameter water jet with a speed of 700 m/s. Determine the flowrate.

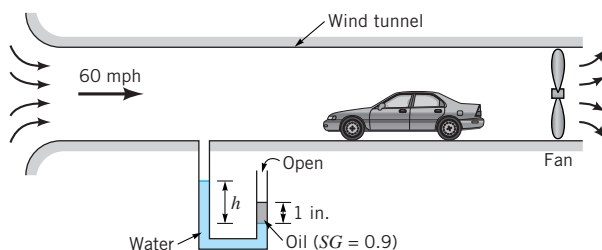


■ FIGURE P3.25



■ FIGURE P3.26

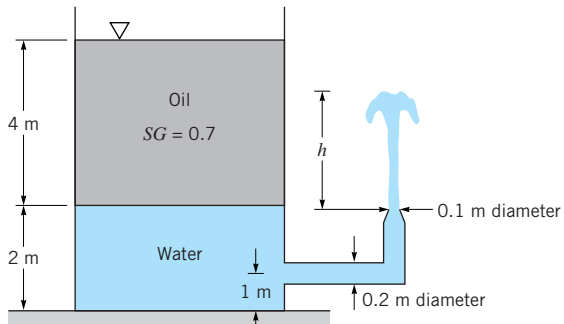
3.27 Air is drawn into a wind tunnel used for testing automobiles as shown in Fig. P3.27. (a) Determine the manometer reading, h , when the velocity in the test section is 60 mph. Note that there is a 1-in. column of oil on the water in the manometer. (b) Determine the difference between the stagnation pressure on the front of the automobile and the pressure in the test section.



■ FIGURE P3.27

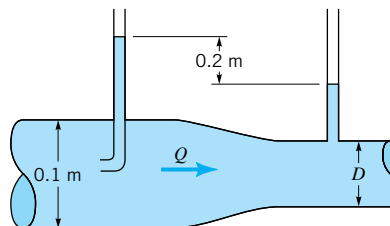
3.28 A loon is a diving bird equally at home “flying” in the air or water. What swimming velocity under water will produce a dynamic pressure equal to that when it flies in the air at 40 mph?

3.29 A large open tank contains a layer of oil floating on water as shown in Fig. P3.29. The flow is steady and inviscid. (a) Determine the height, h , to which the water will rise. (b) Determine the water velocity in the pipe. (c) Determine the pressure in the horizontal pipe.



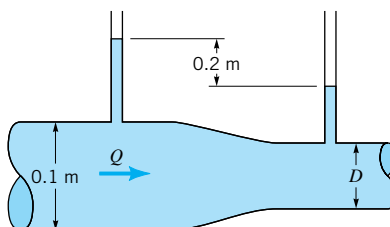
■ FIGURE P3.29

3.30 Water flows through the pipe contraction shown in Fig. P3.30. For the given 0.2-m difference in manometer level, determine the flowrate as a function of the diameter of the small pipe, D .



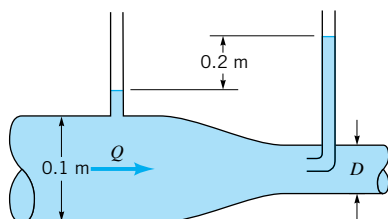
■ FIGURE P3.30

3.31 Water flows through the pipe contraction shown in Fig. P3.31. For the given 0.2-m difference in the manometer level, determine the flowrate as a function of the diameter of the small pipe, D .



■ FIGURE P3.31

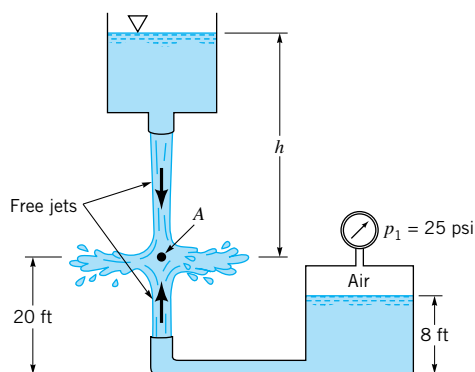
3.32 Water flows through the pipe contraction shown in Fig. P3.32. For the given 0.2-m difference in the manometer level, determine the flowrate as a function of the diameter of the small pipe, D .



■ FIGURE P3.32

3.33 The speed of an airplane through the air is obtained by use of a Pitot-static tube that measures the difference between the stagnation and static pressures. (See Video V3.4.) Rather than indicating this pressure difference (psi or N/m²) directly, the indicator is calibrated in speed (mph or knots). This calibration is done using the density of standard sea level air. Thus, the air speed displayed (termed the indicated air speed) is the actual air speed only at standard sea level conditions. If the aircraft is flying at an altitude of 20,000 ft and the indicated air speed is 220 knots, what is the actual air speed?

3.34 Streams of water from two tanks impinge upon each other as shown in Fig. P3.34. If viscous effects are negligible and point A is a stagnation point, determine the height h .

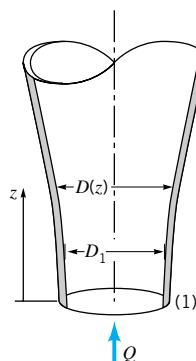


■ FIGURE P3.34

3.35 A 0.15-m-diameter pipe discharges into a 0.10-m-diameter pipe. Determine the velocity head in each pipe if they are carrying 0.12 m³/s of kerosene.

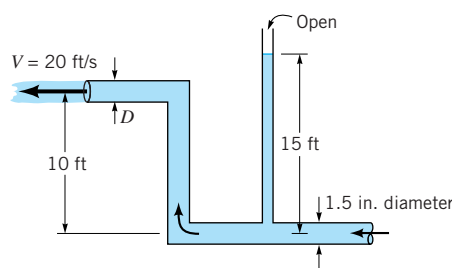
3.36 Water flows upward through a variable area pipe with a constant flowrate, Q , as shown in Fig. P3.36. If viscous effects are negligible, determine the diameter, $D(z)$, in terms of D_1 if the pressure is to remain constant throughout the pipe. That is, $p(z) = p_1$.

3.37 Water flows steadily with negligible viscous effects through the pipe shown in Fig. P3.37. Determine the diameter,



■ FIGURE P3.36

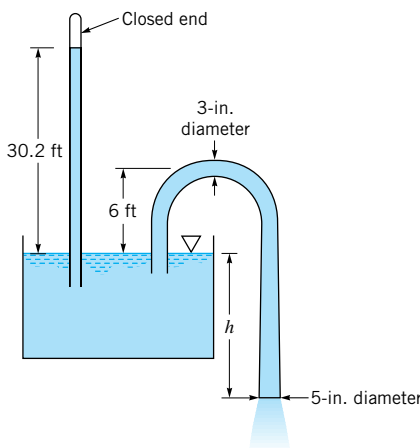
D , of the pipe at the outlet (a free jet) if the velocity there is 20 ft/s.



■ FIGURE P3.37

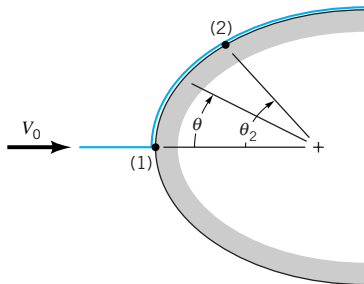
3.38 The circular stream of water from a faucet is observed to taper from a diameter of 20 mm to 10 mm in a distance of 50 cm. Determine the flowrate.

3.39 Water is siphoned from the tank shown in Fig. P3.39. The water barometer indicates a reading of 30.2 ft. Determine the maximum value of h allowed without cavitation occurring. Note that the pressure of the vapor in the closed end of the barometer equals the vapor pressure.



■ FIGURE P3.39

3.40 An inviscid fluid flows steadily along the stagnation streamline shown in Fig. P3.40 and [Video V3.3](#), starting with speed V_0 far upstream of the object. Upon leaving the stagnation point, point (1), the fluid speed along the surface of the object is assumed to be given by $V = 2 V_0 \sin \theta$, where θ is the angle indicated. At what angular position, θ_2 , should a hole be drilled to give a pressure difference of $p_1 - p_2 = \rho V_0^2/2$? Gravity is negligible.

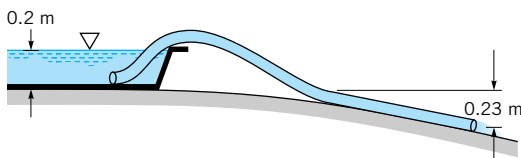


■ FIGURE P3.40

3.41 A certain vacuum cleaner can create a vacuum of 2 kPa just inside the hose. What is the velocity of the air inside the hose?

3.42 Water from a faucet fills a 16-oz glass (volume = 28.9 in.³) in 10 s. If the diameter of the jet leaving the faucet is 0.60 in., what is the diameter of the jet when it strikes the water surface in the glass which is positioned 14 in. below the faucet?

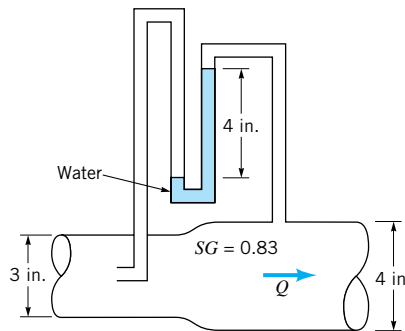
3.43 A smooth plastic, 10-m-long garden hose with an inside diameter of 20 mm is used to drain a wading pool as is shown in Fig. P3.43. If viscous effects are neglected, what is the flowrate from the pool?



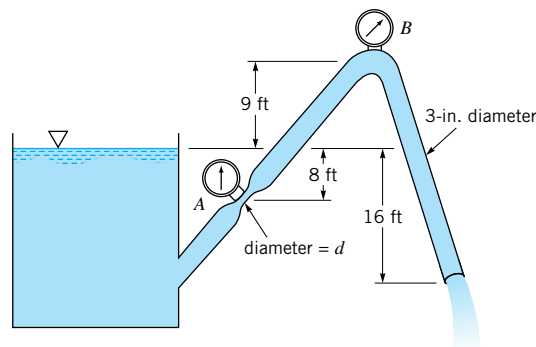
■ FIGURE P3.43

3.44 Carbon dioxide flows at a rate of 1.5 ft³/s from a 3-in. pipe in which the pressure and temperature are 20 psi (gage) and 120 °F into a 1.5-in. pipe. If viscous effects are neglected and incompressible conditions are assumed, determine the pressure in the smaller pipe.

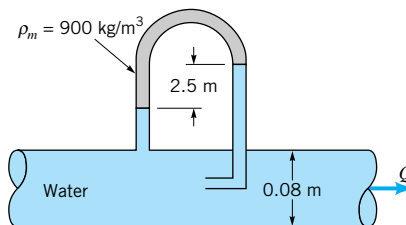
3.45 Oil of specific gravity 0.83 flows in the pipe shown in Fig. P3.45. If viscous effects are neglected, what is the flowrate?



■ FIGURE P3.45

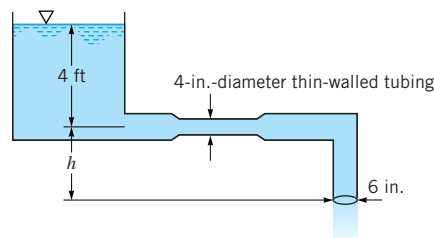


■ FIGURE P3.46



■ FIGURE P3.47

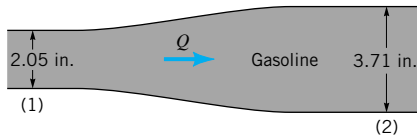
3.48 Water flows steadily with negligible viscous effects through the pipe shown in Fig. P3.48. It is known that the 4-in.-diameter section of thin-walled tubing will collapse if the pressure within it becomes less than 10 psi below atmospheric pressure. Determine the maximum value that h can have without causing collapse of the tubing.



■ FIGURE P3.48

3.47 Determine the flowrate through the pipe in Fig. P3.47.

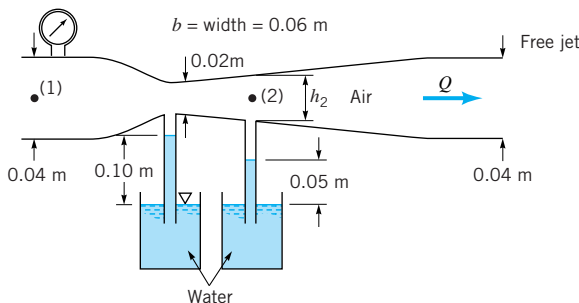
3.49 For the pipe enlargement shown in Fig. P3.49, the pressures at sections (1) and (2) are 56.3 and 58.2 psi, respectively. Determine the weight flowrate (lb/s) of the gasoline in the pipe.



■ FIGURE P3.49

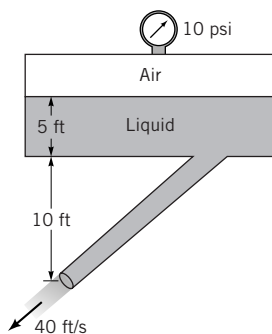
3.50 Water is pumped from a lake through an 8-in. pipe at a rate of $10 \text{ ft}^3/\text{s}$. If viscous effects are negligible, what is the pressure in the suction pipe (the pipe between the lake and the pump) at an elevation 6 ft above the lake?

3.51 Air flows through a Venturi channel of rectangular cross section as shown in Video V3.6 and Fig. P3.51. The constant width of the channel is 0.06 m and the height at the exit is 0.04 m. Compressibility and viscous effects are negligible. (a) Determine the flowrate when water is drawn up 0.10 m in a small tube attached to the static pressure tap at the throat where the channel height is 0.02 m. (b) Determine the channel height, h_2 , at section (2) where, for the same flowrate as in part (a), the water is drawn up 0.05 m. (c) Determine the pressure needed at section (1) to produce this flow.



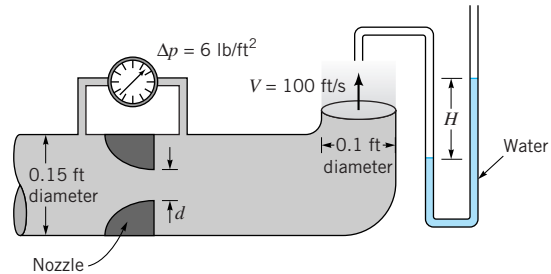
■ FIGURE P3.51

3.52 An inviscid, incompressible liquid flows steadily from the large pressurized tank shown in Fig. P3.52. The velocity at the exit is 40 ft/s. Determine the specific gravity of the liquid in the tank.



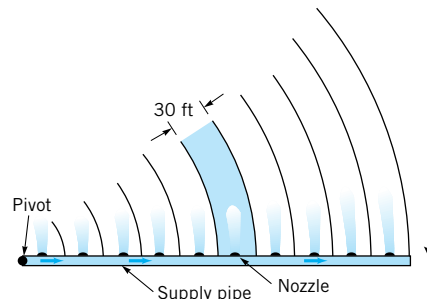
■ FIGURE P3.52

3.53 Air (assumed frictionless and incompressible) flows steadily through the device shown in Fig. P3.53. The exit velocity is 100 ft/s, and the differential pressure across the nozzle is $6 \text{ lb}/\text{ft}^2$. (a) Determine the reading, H , for the water-filled manometer attached to the Pitot tube. (b) Determine the diameter, d , of the nozzle.



■ FIGURE P3.53

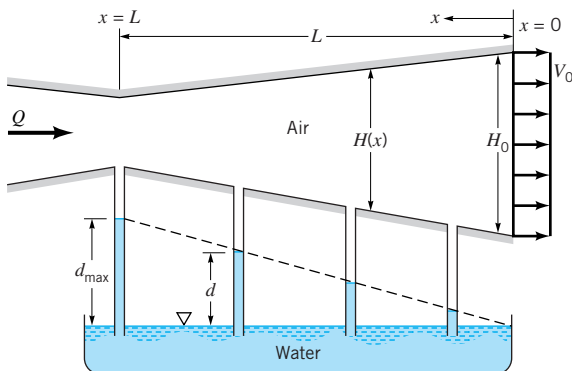
3.54 The center pivot irrigation system shown in Fig. P3.54 is to provide uniform watering of the entire circular field. Water flows through the common supply pipe and out through 10 evenly spaced nozzles. Water from each nozzle is to cover a strip 30 feet wide as indicated. If viscous effects are negligible, determine the diameter of each nozzle, d_i , $i = 1$ to 10, in terms of the diameter, d_{10} , of the nozzle at the outer end of the arm.



■ FIGURE P3.54

3.55 Air flows steadily through a converging–diverging rectangular channel of constant width as shown in Fig. P3.55 and Video V3.6. The height of the channel at the exit and the exit velocity are H_0 and V_0 , respectively. The channel is to be shaped so that the distance, d , that water is drawn up into tubes attached to static pressure taps along the channel wall is linear with distance along the channel. That is, $d = (d_{\max}/L)x$, where L is the channel length and d_{\max} is the maximum water depth (at the minimum channel height; $x = L$). Determine the height, $H(x)$, as a function of x and the other important parameters.

***3.56** Air flows through a horizontal pipe of variable diameter, $D = D(x)$, at a rate of $1.5 \text{ ft}^3/\text{s}$. The static pressure distribution obtained from a set of 12 static pressure taps along the pipe wall is as shown below. Plot the pipe shape, $D(x)$, if the

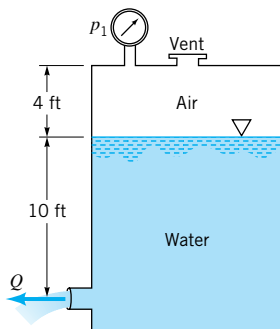


■ FIGURE P3.55

diameter at $x = 0$ is 1, 2, or 3 in. Neglect viscous and compressibility effects.

x (in.)	p (in. H ₂ O)	x (in.)	p (in. H ₂ O)
0	1.00	7	0.44
1	0.72	8	0.51
2	0.16	9	0.65
3	-0.96	10	0.78
4	-0.31	11	0.90
5	0.27	12	1.00
6	0.39		

3.57 The vent on the tank shown in Fig. P3.57 is closed and the tank pressurized to increase the flowrate. What pressure, p_1 , is needed to produce twice the flowrate of that when the vent is open?

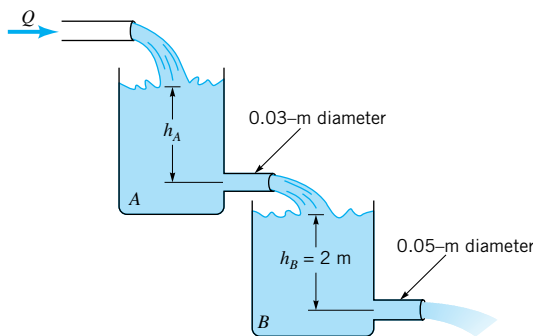


■ FIGURE P3.57

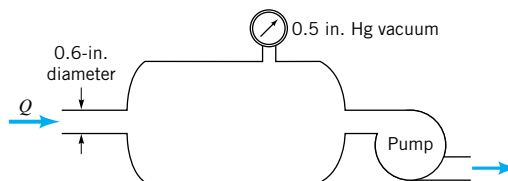
3.58 Water flows steadily through the large tanks shown in Fig. P3.58. Determine the water depth, h_A .

3.59 Air at 80 °F and 14.7 psia flows into the tank shown in Fig. P3.59. Determine the flowrate in ft³/s, lb/s, and slugs/s. Assume incompressible flow.

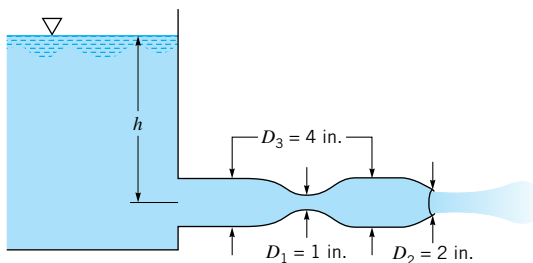
3.60 Water flows from a large tank as shown in Fig. P3.60. Atmospheric pressure is 14.5 psia and the vapor pressure is 1.60 psia. If viscous effects are neglected, at what height, h , will cavitation begin? To avoid cavitation, should the value of D_1 be increased or decreased? To avoid cavitation, should the value of D_2 be increased or decreased? Explain.



■ FIGURE P3.58



■ FIGURE P3.59



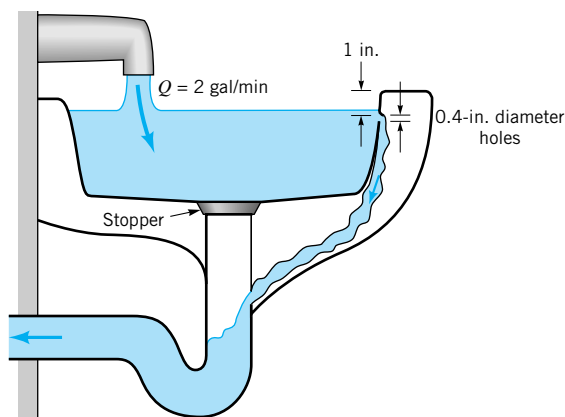
■ FIGURE P3.60

3.61 Water flows into the sink shown in Fig. P3.61 and Video V5.1 at a rate of 2 gal/min. If the drain is closed, the water will eventually flow through the overflow drain holes rather than over the edge of the sink. How many 0.4-in.-diameter drain holes are needed to ensure that the water does not overflow the sink? Neglect viscous effects.

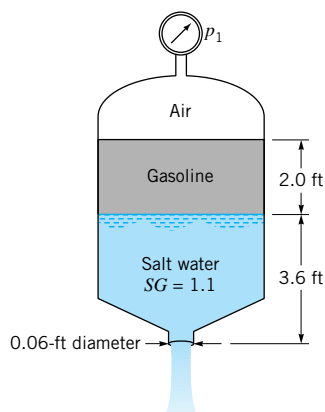
3.62 What pressure, p_1 , is needed to produce a flowrate of 0.09 ft³/s from the tank shown in Fig. P3.62?

3.63 Laboratories containing dangerous materials are often kept at a pressure slightly less than ambient pressure so that contaminants can be filtered through an exhaust system rather than leaked through cracks around doors, etc. If the pressure in such a room is 0.1 in. of water below that of the surrounding rooms, with what velocity will air enter the room through an opening? Assume viscous effects are negligible.

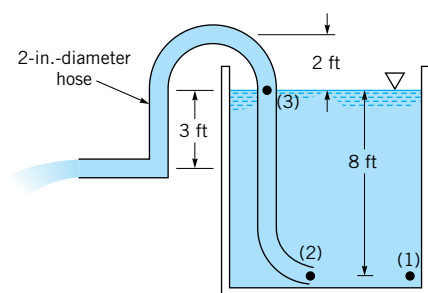
3.64 Water is siphoned from the tank shown in Fig. P3.64. Determine the flowrate from the tank and the pressures at points (1), (2), and (3) if viscous effects are negligible.



■ FIGURE P3.61



■ FIGURE P3.62

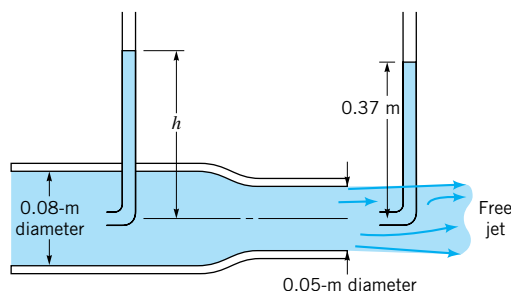


■ FIGURE P3.64

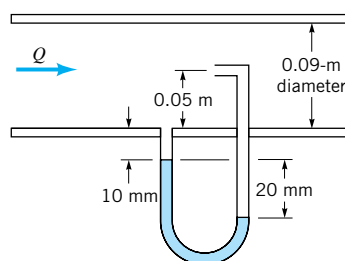
3.65 Redo Problem 3.64 if a 1-in.-diameter nozzle is placed at the end of the tube.

3.66 Determine the manometer reading, h , for the flow shown in Fig. P3.66.

3.67 The specific gravity of the manometer fluid shown in Fig. P3.67 is 1.07. Determine the volume flowrate, Q , if the flow is inviscid and incompressible and the flowing fluid is (a) water, (b) gasoline, or (c) air at standard conditions.

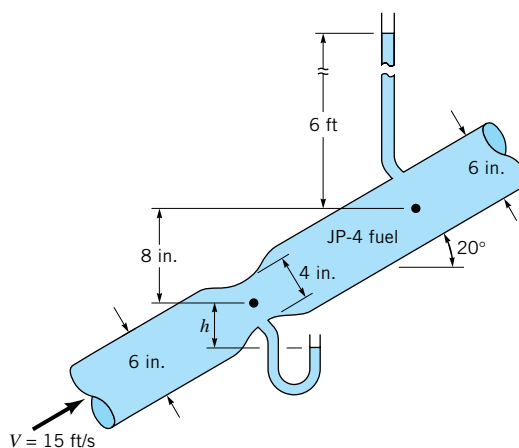


■ FIGURE P3.66



■ FIGURE P3.67

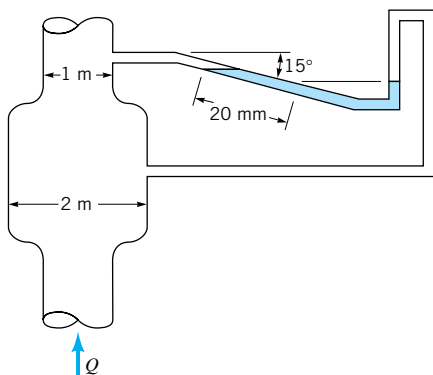
3.68 JP-4 fuel ($SG = 0.77$) flows through the Venturi meter shown in Fig. P3.68 with a velocity of 15 ft/s in the 6-in. pipe. If viscous effects are negligible, determine the elevation, h , of the fuel in the open tube connected to the throat of the Venturi meter.



■ FIGURE P3.68

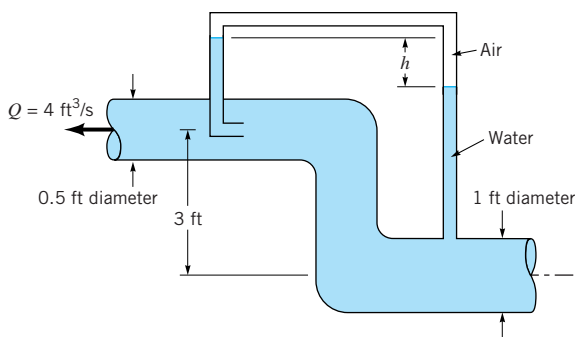
3.69 Repeat Problem 3.68 if the flowing fluid is water rather than JP-4 fuel.

3.70 Air at standard conditions flows through the cylindrical drying stack shown in Fig. P3.70. If viscous effects are negligible and the inclined water-filled manometer reading is 20 mm as indicated, determine the flowrate.



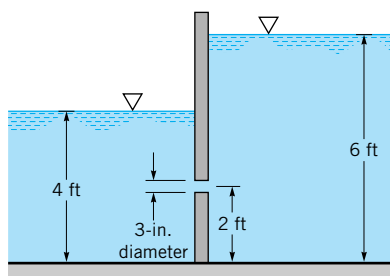
■ FIGURE P3.70

3.71 Water, considered an inviscid, incompressible fluid, flows steadily as shown in Fig. P3.71. Determine h .



■ FIGURE P3.71

3.72 Determine the flowrate through the submerged orifice shown in Fig. P3.72 if the contraction coefficient is $C_c = 0.63$.

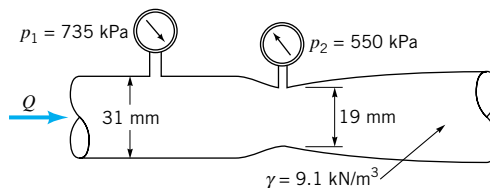


■ FIGURE P3.72

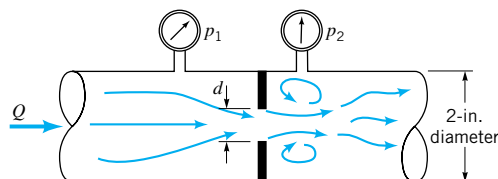
3.73 Determine the flowrate through the Venturi meter shown in Fig. P3.73 if ideal conditions exist.

3.74 For what flowrate through the Venturi meter of Prob. 3.73 will cavitation begin if $p_1 = 275$ kPa gage, atmospheric pressure is 101 kPa (abs), and the vapor pressure is 3.6 kPa (abs)?

3.75 What diameter orifice hole, d , is needed if under ideal conditions the flowrate through the orifice meter of Fig. P3.75 is to be 30 gal/min of seawater with $p_1 - p_2 = 2.37$ lb/in.²? The contraction coefficient is assumed to be 0.63.

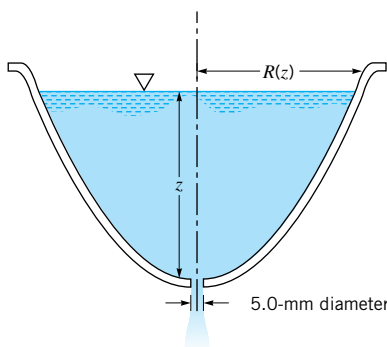


■ FIGURE P3.73



■ FIGURE P3.75

3.76 An ancient device for measuring time is shown in Fig. P3.76. The axisymmetric vessel is shaped so that the water level falls at a constant rate. Determine the shape of the vessel, $R = R(z)$, if the water level is to decrease at a rate of 0.10 m/hr and the drain hole is 5.0 mm in diameter. The device is to operate for 12 hr without needing refilling. Make a scale drawing of the shape of the vessel.



■ FIGURE P3.76

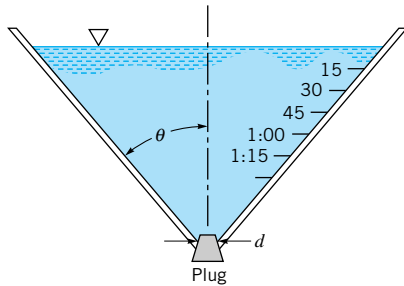
† **3.77** A small hole develops in the bottom of the stationary rowboat shown in Fig. P3.77. Estimate the amount of time it will take for the boat to sink. List all assumptions and show all calculations.



■ FIGURE P3.77

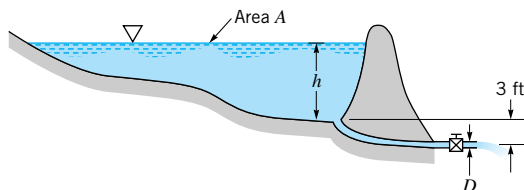
***3.78** A spherical tank of diameter D has a drain hole of diameter d at its bottom. A vent at the top of the tank maintains atmospheric pressure at the liquid surface within the tank. The flow is quasisteady and inviscid and the tank is full of water initially. Determine the water depth as a function of time, $h = h(t)$, and plot graphs of $h(t)$ for tank diameters of 1, 5, 10, and 20 ft if $d = 1$ in.

***3.79** An inexpensive timer is to be made from a funnel as indicated in Fig. P3.79. The funnel is filled to the top with water and the plug is removed at time $t = 0$ to allow the water to run out. Marks are to be placed on the wall of the funnel indicating the time in 15-s intervals, from 0 to 3 min (at which time the funnel becomes empty). If the funnel outlet has a diameter of $d = 0.1$ in, draw to scale the funnel with the timing marks for funnels with angles of $\theta = 30, 45$, and 60° . Repeat the problem if the diameter is changed to 0.05 in.



■ FIGURE P3.79

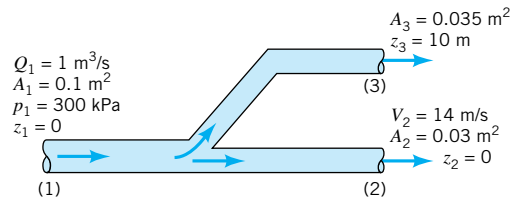
***3.80** The surface area, A , of the pond shown in Fig. P3.80 varies with the water depth, h , as shown in the table. At time $t = 0$ a valve is opened and the pond is allowed to drain through a pipe of diameter D . If viscous effects are negligible and quasi-steady conditions are assumed, plot the water depth as a function of time from when the valve is opened ($t = 0$) until the pond is drained for pipe diameters of $D = 0.5, 1.0, 1.5, 2.0, 2.5$, and 3.0 ft. Assume $h = 18$ ft at $t = 0$.



■ FIGURE P3.80

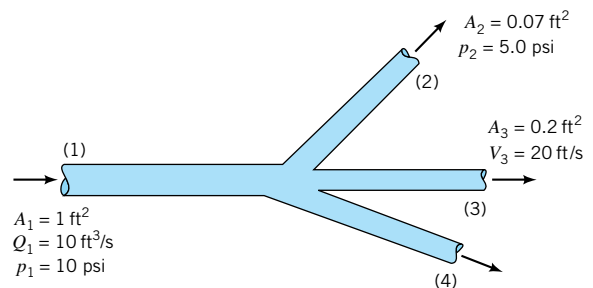
h (ft)	A [acres (1 acre = 43,560 ft ²)]
0	0
2	0.3
4	0.5
6	0.8
8	0.9
10	1.1
12	1.5
14	1.8
16	2.4
18	2.8

3.81 Water flows through the branching pipe shown in Fig. P3.81. If viscous effects are negligible, determine the pressure at section (2) and the pressure at section (3).



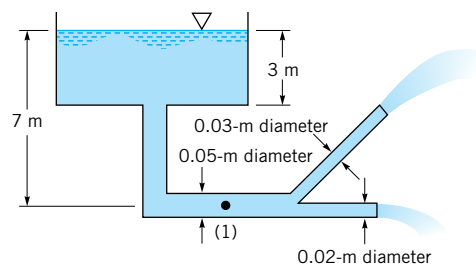
■ FIGURE P3.81

3.82 Water flows through the horizontal branching pipe shown in Fig. P3.82 at a rate of 10 ft³/s. If viscous effects are negligible, determine the water speed at section (2), the pressure at section (3), and the flowrate at section (4).



■ FIGURE P3.82

3.83 Water flows from a large tank through a large pipe that splits into two smaller pipes as shown in Fig. P3.83. If viscous effects are negligible, determine the flowrate from the tank and the pressure at point (1).

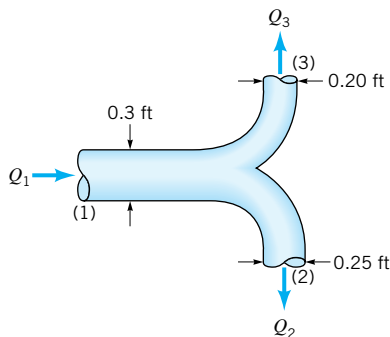


■ FIGURE P3.83

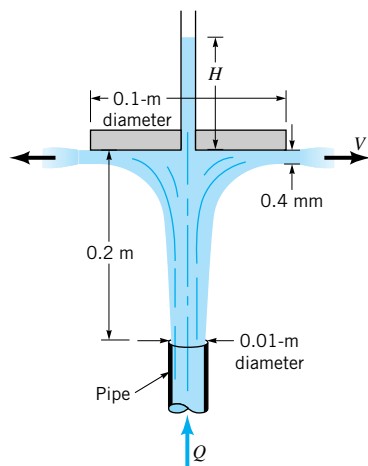
3.84 Water flows through the horizontal Y-fitting shown in Fig. P3.84. If the flowrate and pressure in pipe (1) are $Q_1 = 2.3$ ft³/s and $p_1 = 50$ lb/in.², determine the pressures, p_2 and p_3 , in pipes (2) and (3) under the assumption that the flowrate divides evenly between pipes (2) and (3).

3.85 Water flows from the pipe shown in Fig. P3.85 as a free jet and strikes a circular flat plate. The flow geometry shown is axisymmetrical. Determine the flowrate and the manometer reading, H .

3.86 Air, assumed incompressible and inviscid, flows into the outdoor cooking grill through nine holes of 0.40-in. diameter as shown in Fig. P3.86. If a flowrate of 40 in.³/s into the

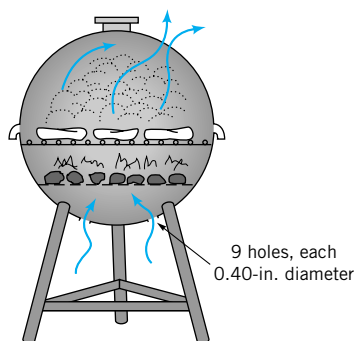


■ FIGURE P3.84



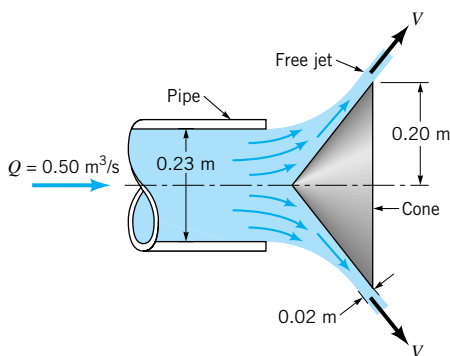
■ FIGURE P3.85

grill is required to maintain the correct cooking conditions, determine the pressure within the grill near the holes.



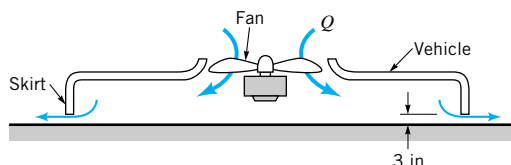
■ FIGURE P3.86

3.87 A conical plug is used to regulate the air flow from the pipe shown in Fig. P3.87. The air leaves the edge of the cone with a uniform thickness of 0.02 m. If viscous effects are negligible and the flowrate is $0.50 \text{ m}^3/\text{s}$, determine the pressure within the pipe.



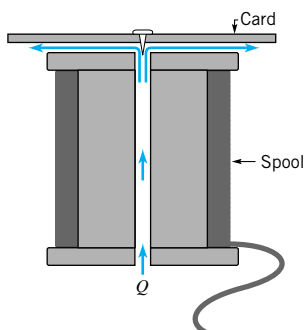
■ FIGURE P3.87

3.88 An air cushion vehicle is supported by forcing air into the chamber created by a skirt around the periphery of the vehicle as shown in Fig. P3.88. The air escapes through the 3-in. clearance between the lower end of the skirt and the ground (or water). Assume the vehicle weighs 10,000 lb and is essentially rectangular in shape, 30 by 50 ft. The volume of the chamber is large enough so that the kinetic energy of the air within the chamber is negligible. Determine the flowrate, Q , needed to support the vehicle. If the ground clearance were reduced to 2 in., what flowrate would be needed? If the vehicle weight were reduced to 5000 lb and the ground clearance maintained at 3 in., what flowrate would be needed?



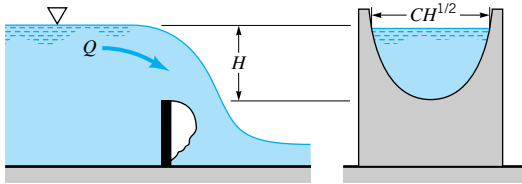
■ FIGURE P3.88

3.89 A small card is placed on top of a spool as shown in Fig. P3.89. It is not possible to blow the card off the spool by blowing air through the hole in the center of the spool. The harder one blows, the harder the card “sticks” to the spool. In fact, by blowing hard enough it is possible to keep the card against the spool with the spool turned upside down. (*Note:* It may be necessary to use a thumb tack to prevent the card from sliding from the spool.) Explain this phenomenon.



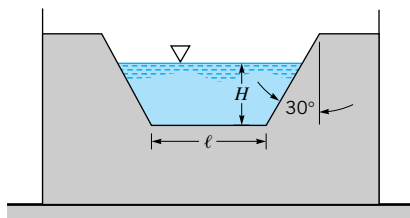
■ FIGURE P3.89

3.90 Water flows over a weir plate (see [Video V10.7](#)) which has a parabolic opening as shown in Fig. P3.90. That is, the opening in the weir plate has a width $CH^{1/2}$, where C is a constant. Determine the functional dependence of the flowrate on the head, $Q = Q(H)$.



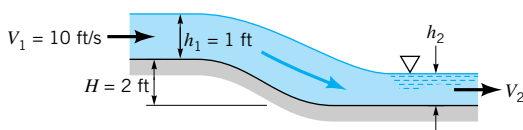
■ FIGURE P3.90

3.91 A weir (see [Video V10.7](#)) of trapezoidal cross section is used to measure the flowrate in a channel as shown in Fig. P3.91. If the flowrate is Q_0 when $H = \ell/2$, what flowrate is expected when $H = \ell$?



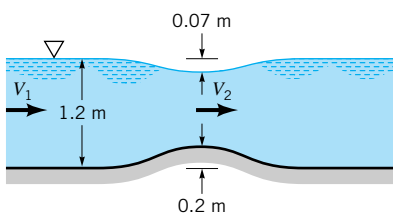
■ FIGURE P3.91

3.92 Water flows down the sloping ramp shown in Fig. P3.92 with negligible viscous effects. The flow is uniform at sections (1) and (2). For the conditions given, show that three solutions for the downstream depth, h_2 , are obtained by use of the Bernoulli and continuity equations. However, show that only two of these solutions are realistic. Determine these values.



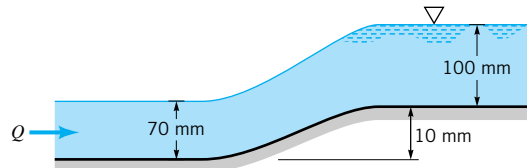
■ FIGURE P3.92

3.93 The flowrate in a water channel is sometimes determined by use of a device called a Venturi flume. As shown in Fig. P3.93, this device consists simply of a hump on the bottom of the channel. If the water surface dips a distance of 0.07 m for the conditions shown, what is the flowrate per width of the channel? Assume the velocity is uniform and viscous effects are negligible.



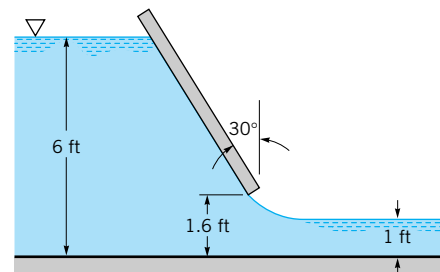
■ FIGURE P3.93

3.94 Water flows in a rectangular channel that is 2.0 m wide as shown in Fig. P3.94. The upstream depth is 70 mm. The water surface rises 40 mm as it passes over a portion where the channel bottom rises 10 mm. If viscous effects are negligible, what is the flowrate?



■ FIGURE P3.94

3.95 Water flows under the inclined sluice gate shown in Fig. P3.95. Determine the flowrate if the gate is 8 ft wide.



■ FIGURE P3.95

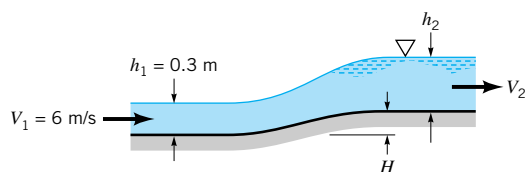
3.96 Water flows in a vertical pipe of 0.15-m diameter at a rate of $0.2 \text{ m}^3/\text{s}$ and a pressure of 200 kPa at an elevation of 25 m. Determine the velocity head and pressure head at elevations of 20 and 55 m.

3.97 Draw the energy line and the hydraulic grade line for the flow shown in Problem 3.64.

3.98 Draw the energy line and the hydraulic grade line for the flow of Problem 3.60.

3.99 Draw the energy line and hydraulic grade line for the flow shown in Problem 3.65.

***3.100** Water flows up the ramp shown in Fig. P3.100 with negligible viscous losses. The upstream depth and velocity are maintained at $h_1 = 0.3 \text{ m}$ and $V_1 = 6 \text{ m/s}$. Plot a graph of the downstream depth, h_2 , as a function of the ramp height, H , for $0 \leq H \leq 2 \text{ m}$. Note that for each value of H there are three solutions, not all of which are realistic.



■ FIGURE P3.100

3.101 This problem involves the pressure distribution between two parallel circular plates. To proceed with this problem, [click here](#) in the E-book.

3.102 This problem involves the calibration of a nozzle-type flow meter. To proceed with this problem, [click here](#) in the E-book.

3.103 This problem involves the pressure distribution in a two-dimensional channel. To proceed with this problem, [click here](#) in the E-book.

3.104 This problem involves the determination of the flowrate under a sluice gate as a function of the water depth. To proceed with this problem, [click here](#) in the E-book.