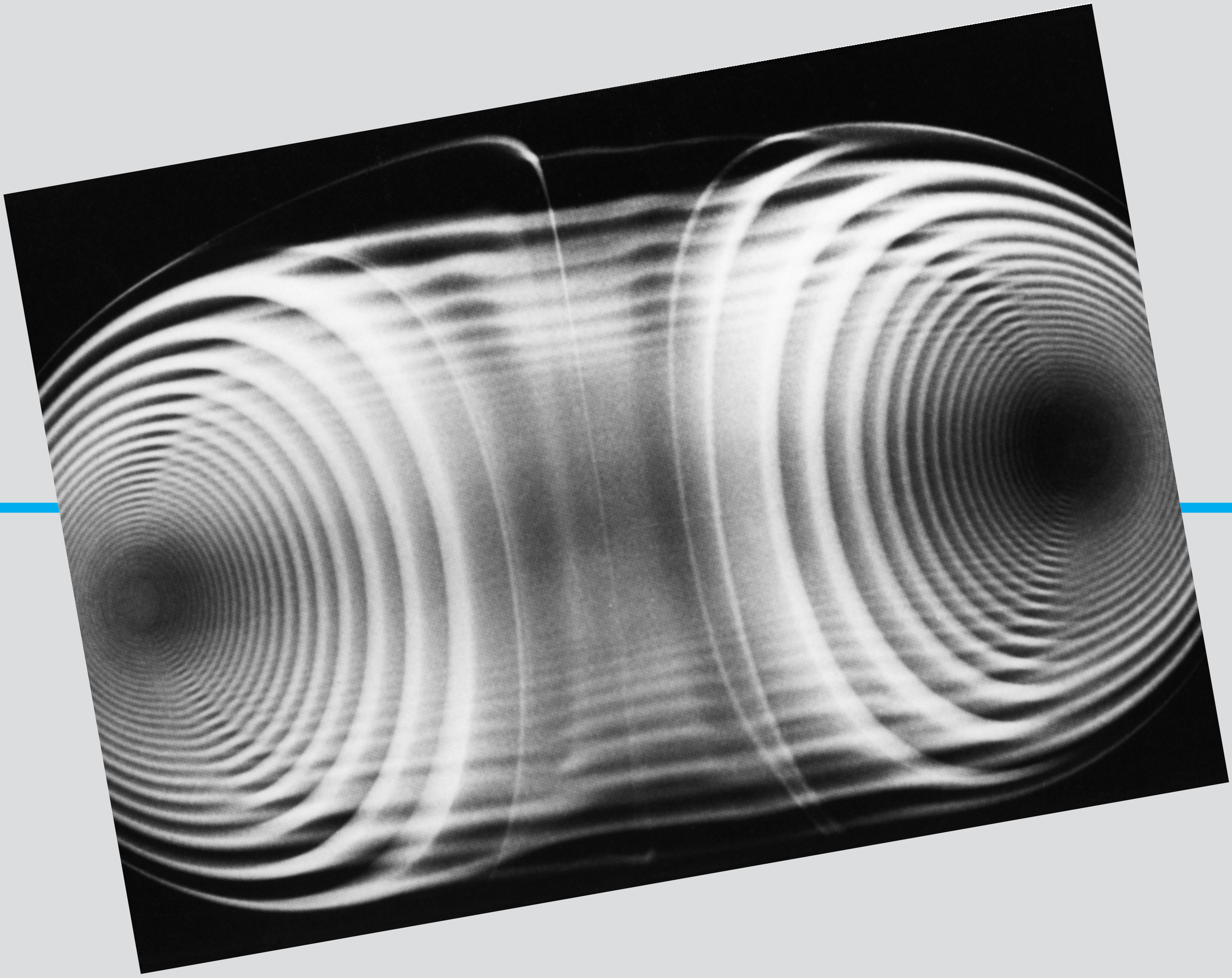
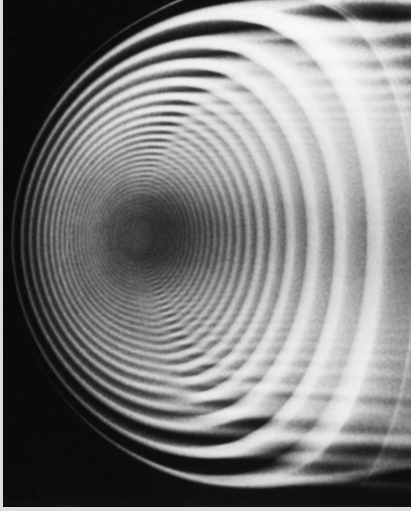


A vortex ring: The complex, three-dimensional structure of a smoke ring is indicated in this cross-sectional view. (Smoke in air.) (Photograph courtesy of R. H. Magarvey and C. S. MacLatchy, Ref. 4.)

4 Fluid Kinematics



In the previous three chapters we have defined some basic properties of fluids and have considered various situations involving fluids that are either at rest or are moving in a rather elementary manner. In general, fluids have a well-known tendency to move or flow. It is very difficult to "tie down" a fluid and restrain it from moving. The slightest of shear stresses will cause the fluid to move. Similarly, an appropriate imbalance of normal stresses (pressure) will cause fluid motion.

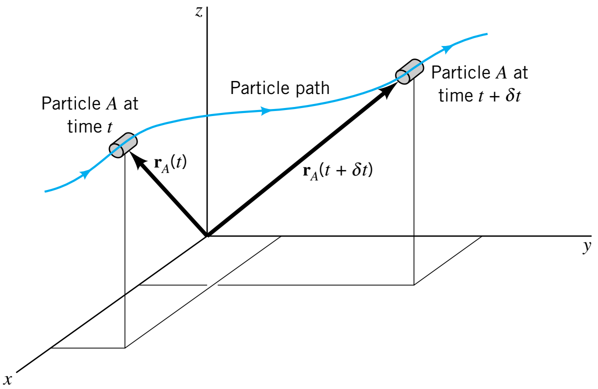
Kinematics involves position, velocity, and acceleration, not force.

In this chapter we will discuss various aspects of fluid motion without being concerned with the actual forces necessary to produce the motion. That is, we will consider the *kinematics* of the motion—the velocity and acceleration of the fluid, and the description and visualization of its motion. The analysis of the specific forces necessary to produce the motion (the *dynamics* of the motion) will be discussed in detail in the following chapters. A wide variety of useful information can be gained from a thorough understanding of fluid kinematics. Such an understanding of how to describe and observe fluid motion is an essential step to the complete understanding of fluid dynamics.

We have all observed fascinating fluid motions like those associated with the smoke emerging from a chimney or the flow of the atmosphere as indicated by the motion of clouds. The motion of waves on a lake or the mixing of paint in a bucket provide other common, although quite different, examples of flow visualization. Considerable insight into these fluid motions can be gained by considering the kinematics of such flows without being concerned with the specific force that drives them.

4.1 The Velocity Field

In general, fluids flow. That is, there is a net motion of molecules from one point in space to another point as a function of time. As is discussed in Chapter 1, a typical portion of fluid contains so many molecules that it becomes totally unrealistic (except in special cases) for us to attempt to account for the motion of individual molecules. Rather, we employ the continuum hypothesis and consider fluids to be made up of fluid particles that interact with each other and



■ FIGURE 4.1 **Particle location in terms of its position vector.**

with their surroundings. Each particle contains numerous molecules. Thus, we can describe the flow of a fluid in terms of the motion of fluid particles rather than individual molecules. This motion can be described in terms of the velocity and acceleration of the fluid particles.

The infinitesimal particles of a fluid are tightly packed together (as is implied by the continuum assumption). Thus, at a given instant in time, a description of any fluid property (such as density, pressure, velocity, and acceleration) may be given as a function of the fluid's location. This representation of fluid parameters as functions of the spatial coordinates is termed a *field representation* of the flow. Of course, the specific field representation may be different at different times, so that to describe a fluid flow we must determine the various parameters not only as a function of the spatial coordinates (x, y, z, for example) but also as a function of time, t. Thus, to completely specify the temperature, T, in a room we must specify the temperature field, $T = T(x, y, z, t)$, throughout the room (from floor to ceiling and wall to wall) at any time of the day or night.

Fluid parameters can be described by a field representation.

One of the most important fluid variables is the *velocity field,*

$$\mathbf{V} = u(x, y, z, t)\hat{\mathbf{i}} + v(x, y, z, t)\hat{\mathbf{j}} + w(x, y, z, t)\hat{\mathbf{k}}$$

where u, v, and w are the x, y, and z components of the velocity vector. By definition, the velocity of a particle is the time rate of change of the position vector for that particle. As is illustrated in Fig. 4.1, the position of particle A relative to the coordinate system is given by its *position vector*, $\mathbf{r}_A$, which (if the particle is moving) is a function of time. The time derivative of this position gives the *velocity* of the particle, $d\mathbf{r}_A/dt = \mathbf{V}_A$. By writing the velocity for all of the particles we can obtain the field description of the velocity vector $\mathbf{V} = \mathbf{V}(x, y, z, t)$.

V4.1 Velocity field

Since the velocity is a vector, it has both a direction and a magnitude. The magnitude of $\mathbf{V}$, denoted $V = |\mathbf{V}| = (u^2 + v^2 + w^2)^{1/2}$, is the speed of the fluid. (It is very common in practical situations to call V velocity rather than speed, i.e., "the velocity of the fluid is 12 m/s.") As is discussed in the next section, a change in velocity results in an acceleration. This acceleration may be due to a change in speed and/or direction.

EXAMPLE 4.1

A velocity field is given by $\mathbf{V} = (V_0/\ell)(x\hat{\mathbf{i}} - y\hat{\mathbf{j}})$ where V_0 and ℓ are constants. At what location in the flow field is the speed equal to V_0? Make a sketch of the velocity field in the first quadrant ($x \geq 0$, $y \geq 0$) by drawing arrows representing the fluid velocity at representative locations.

SOLUTION

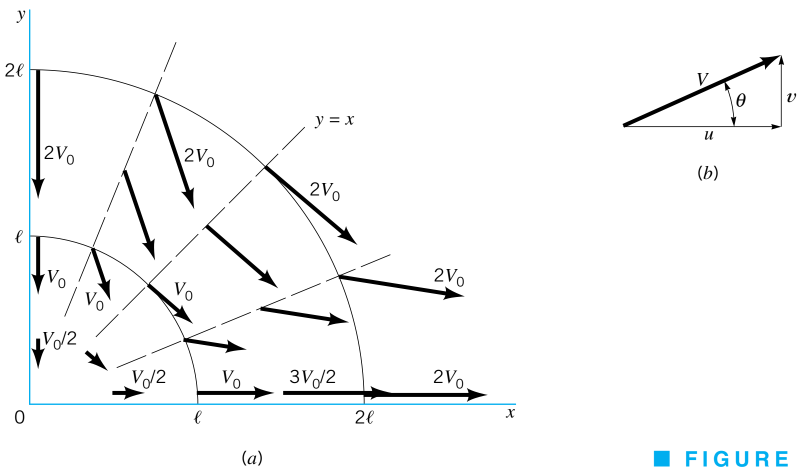
The x, y, and z components of the velocity are given by $u = V_0 x/\ell$, $v = -V_0 y/\ell$, and $w = 0$ so that the fluid speed, V, is

$$V = (u^2 + v^2 + w^2)^{1/2} = \frac{V_0}{\ell}(x^2 + y^2)^{1/2} \tag{1}$$

The speed is $V = V_0$ at any location on the circle of radius ℓ centered at the origin $[(x^2 + y^2)^{1/2} = \ell]$ as shown in Fig. E4.1*a*. **(Ans)**

The direction of the fluid velocity relative to the x axis is given in terms of $\theta = \arctan(v/u)$ as shown in Fig. E4.1*b*. For this flow

$$\tan\theta = \frac{v}{u} = \frac{-V_0 y/\ell}{V_0 x/\ell} = \frac{-y}{x}$$



■ FIGURE E4.1

Thus, along the x axis ($y = 0$) we see that $\tan\theta = 0$, so that $\theta = 0°$ or $\theta = 180°$. Similarly, along the y axis ($x = 0$) we obtain $\tan\theta = \pm\infty$ so that $\theta = 90°$ or $\theta = 270°$. Also, for $y = 0$ we find $\mathbf{V} = (V_0 x/\ell)\hat{\mathbf{i}}$, while for $x = 0$ we have $\mathbf{V} = (-V_0 y/\ell)\hat{\mathbf{j}}$, indicating (if $V_0 > 0$) that the flow is directed toward the origin along the y axis and away from the origin along the x axis as shown in Fig. E4.1*a*.

By determining $\mathbf{V}$ and θ for other locations in the x–y plane, the velocity field can be sketched as shown in the figure. For example, on the line $y = x$ the velocity is at a $-45°$ angle relative to the x axis ($\tan\theta = v/u = -y/x = -1$). At the origin $x = y = 0$ so that $\mathbf{V} = 0$. This point is a stagnation point. The farther from the origin the fluid is, the faster it is flowing (as seen from Eq. 1). By careful consideration of the velocity field it is possible to determine considerable information about the flow.

4.1.1 Eulerian and Lagrangian Flow Descriptions

Either Eulerian or Lagrangian methods can be used to describe flow fields.

There are two general approaches in analyzing fluid mechanics problems (or problems in other branches of the physical sciences, for that matter). The first method, called the *Eulerian method*, uses the field concept introduced above. In this case, the fluid motion is given by completely prescribing the necessary properties (pressure, density, velocity, etc.) as functions of space and time. From this method we obtain information about the flow in terms of what happens at fixed points in space as the fluid flows past those points.

The second method, called the *Lagrangian method*, involves following individual fluid particles as they move about and determining how the fluid properties associated with these particles change as a function of time. That is, the fluid particles are "tagged" or identified, and their properties determined as they move.

The difference between the two methods of analyzing fluid flow problems can be seen in the example of smoke discharging from a chimney, as is shown in Fig. 4.2. In the Eulerian method one may attach a temperature-measuring device to the top of the chimney (point 0) and record the temperature at that point as a function of time. At different times there are different fluid particles passing by the stationary device. Thus, one would obtain the temperature, T, for that location ($x = x_0$, $y = y_0$, and $z = z_0$) as a function of time. That is, $T = T(x_0, y_0, z_0, t)$. The use of numerous temperature-measuring devices fixed at various locations would provide the temperature field, $T = T(x, y, z, t)$. The temperature of a particle as a function of time would not be known unless the location of the particle were known as a function of time.

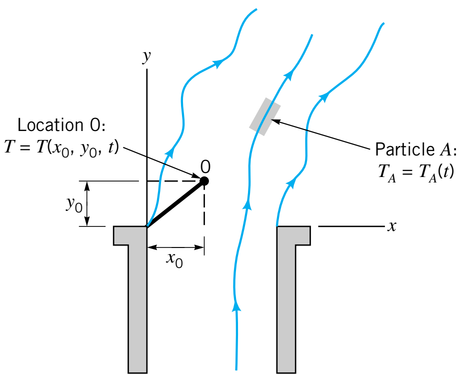
In the Lagrangian method, one would attach the temperature-measuring device to a particular fluid particle (particle A) and record that particle's temperature as it moves about. Thus, one would obtain that particle's temperature as a function of time, $T_A = T_A(t)$. The use of many such measuring devices moving with various fluid particles would provide the temperature of these fluid particles as a function of time. The temperature would not be known as a function of position unless the location of each particle were known as a function of time. If enough information in Eulerian form is available, Lagrangian information can be derived from the Eulerian data—and vice versa.

Example 4.1 provides an Eulerian description of the flow. For a Lagrangian description we would need to determine the velocity as a function of time for each particle as it flows along from one point to another.

Most fluid mechanics considerations involve the Eulerian method.

In fluid mechanics it is usually easier to use the Eulerian method to describe a flow—in either experimental or analytical investigations. There are, however, certain instances in which the Lagrangian method is more convenient. For example, some numerical fluid mechanics calculations are based on determining the motion of individual fluid particles (based on the appropriate interactions among the particles), thereby describing the motion in Lagrangian terms. Similarly, in some experiments individual fluid particles are "tagged" and are followed throughout their motion, providing a Lagrangian description. Oceanographic measurements obtained from devices that flow with the ocean currents provide this information. Similarly, by using X-ray opaque dyes it is possible to trace blood flow in arteries and to obtain a Lagrangian description of the fluid motion. A Lagrangian description may also be useful in describing fluid machinery (such as pumps and turbines) in which fluid particles gain or lose energy as they move along their flow paths.

Another illustration of the difference between the Eulerian and Lagrangian descriptions can be seen in the following biological example. Each year thousands of birds migrate between their summer and winter habitats. Ornithologists study these migrations to obtain various types of important information. One set of data obtained is the rate at which birds pass a cer-



■ **FIGURE 4.2 Eulerian and Lagrangian descriptions of temperature of a flowing fluid.**

tain location on their migration route (birds per hour). This corresponds to an Eulerian description—"flowrate" at a given location as a function of time. Individual birds need not be followed to obtain this information. Another type of information is obtained by "tagging" certain birds with radio transmitters and following their motion along the migration route. This corresponds to a Lagrangian description—"position" of a given particle as a function of time.

4.1.2 One-, Two-, and Three-Dimensional Flows

Generally, a fluid flow is a rather complex three-dimensional, time-dependent phenomenon—$\mathbf{V} = \mathbf{V}(x, y, z, t) = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}}$. In many situations, however, it is possible to make simplifying assumptions that allow a much easier understanding of the problem without sacrificing needed accuracy. One of these simplifications involves approximating a real flow as a simpler one- or two-dimensional flow.

Most flow fields are actually three-dimensional.

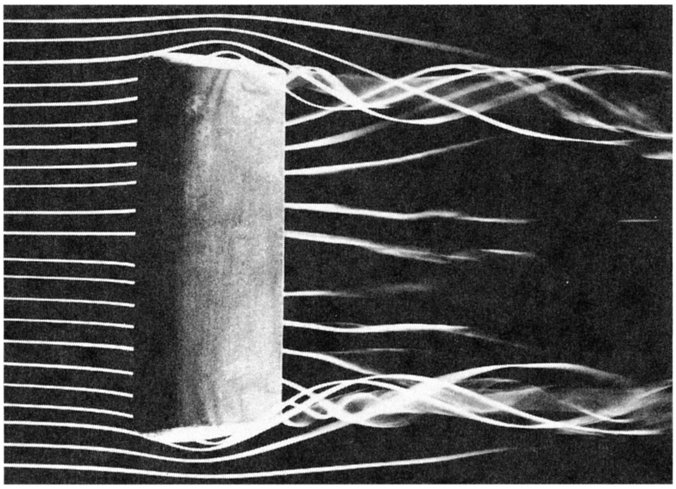
In almost any flow situation, the velocity field actually contains all three velocity components (u, v, and w, for example). In many situations the *three-dimensional flow* characteristics are important in terms of the physical effects they produce. (See the photograph at the beginning of **Chapter 4**.) For these situations it is necessary to analyze the flow in its complete three-dimensional character. Neglect of one or two of the velocity components in these cases would lead to considerable misrepresentation of the effects produced by the actual flow.

V4.2 Flow past a wing

The flow of air past an airplane wing provides an example of a complex three-dimensional flow. A feel for the three-dimensional structure of such flows can be obtained by studying Fig. 4.3, which is a photograph of the flow past a model airfoil; the flow has been made visible by using a flow visualization technique.

In many situations one of the velocity components may be small (in some sense) relative to the two other components. In situations of this kind it may be reasonable to neglect the smaller component and assume *two-dimensional flow*. That is, $\mathbf{V} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}}$, where u and v are functions of x and y (and possibly time, t).

It is sometimes possible to further simplify a flow analysis by assuming that two of the velocity components are negligible, leaving the velocity field to be approximated as a *one-dimensional flow* field. That is, $\mathbf{V} = u\hat{\mathbf{i}}$. As we will learn from examples throughout the remainder of the book, although there are very few, if any, flows that are truly one-dimensional, there are many flow fields for which the one-dimensional flow assumption provides a reasonable approximation. There are also many flow situations for which use of a one-dimensional flow field assumption will give completely erroneous results.



■ **FIGURE 4.3**
Flow visualization of the complex three-dimensional flow past a model airfoil. (Photograph by M. R. Head.)

4.1.3 Steady and Unsteady Flows

In the previous discussion we have assumed *steady flow*—the velocity at a given point in space does not vary with time, $\partial \mathbf{V}/\partial t = 0$. In reality, almost all flows are unsteady in some sense. That is, the velocity does vary with time. It is not difficult to believe that *unsteady flows* are usually more difficult to analyze (and to investigate experimentally) than are steady flows. Hence, considerable simplicity often results if one can make the assumption of steady flow without compromising the usefulness of the results. Among the various types of unsteady flows are nonperiodic flow, periodic flow, and truly random flow. Whether or not unsteadiness of one or more of these types must be included in an analysis is not always immediately obvious.

An example of a nonperiodic, unsteady flow is that produced by turning off a faucet to stop the flow of water. Usually this unsteady flow process is quite mundane and the forces developed as a result of the unsteady effects need not be considered. However, if the water is turned off suddenly (as with an electrically operated valve in a dishwasher), the unsteady effects can become important [as in the "water hammer" effects made apparent by the loud banging of the pipes under such conditions (Ref. 1)].

In other flows the unsteady effects may be periodic, occurring time after time in basically the same manner. The periodic injection of the air-gasoline mixture into the cylinder of an automobile engine is such an example. The unsteady effects are quite regular and repeatable in a regular sequence. They are very important in the operation of the engine.

V4.3 Flow types

In many situations the unsteady character of a flow is quite random. That is, there is no repeatable sequence or regular variation to the unsteadiness. This behavior occurs in *turbulent flow* and is absent from *laminar flow.* The "smooth" flow of highly viscous syrup onto a pancake represents a "deterministic" laminar flow. It is quite different from the turbulent flow observed in the "irregular" splashing of water from a faucet onto the sink below it. The "irregular" gustiness of the wind represents another random turbulent flow. The differences between these types of flows are discussed in considerable detail in **Chapters 8** and **9**.

V4.4 Jupiter red spot

It must be understood that the definition of steady or unsteady flow pertains to the behavior of a fluid property as observed at a fixed point in space. For steady flow, the values of all fluid properties (velocity, temperature, density, etc.) at any fixed point are independent of time. However, the value of those properties for a given fluid particle may change with time as the particle flows along, even in steady flow. Thus, the temperature of the exhaust at the exit of a car's exhaust pipe may be constant for several hours, but the temperature of a fluid particle that left the exhaust pipe five minutes ago is lower now than it was when it left the pipe, even though the flow is steady.

4.1.4 Streamlines, Streaklines, and Pathlines

Although fluid motion can be quite complicated, there are various concepts that can be used to help in the visualization and analysis of flow fields. To this end we discuss the use of streamlines, streaklines, and pathlines in flow analysis. The streamline is often used in analytical work while the streakline and pathline are often used in experimental work.

Streamlines are lines tangent to the velocity field.

A *streamline* is a line that is everywhere tangent to the velocity field. If the flow is steady, nothing at a fixed point (including the velocity direction) changes with time, so the streamlines are fixed lines in space. (See the photograph at the beginning of **Chapter 6**.) For unsteady flows the streamlines may change shape with time. Streamlines are obtained analytically by integrating the equations defining lines tangent to the velocity field. For two-dimensional flows the slope of the streamline, dy/dx, must be equal to the tangent of the angle that the velocity vector makes with the x axis or

$$\frac{dy}{dx} = \frac{v}{u} \tag{4.1}$$

If the velocity field is known as a function of x and y (and t if the flow is unsteady), this equation can be integrated to give the equation of the streamlines.

For unsteady flow there is no easy way to produce streamlines experimentally in the laboratory. As discussed below, the observation of dye, smoke, or some other tracer injected into a flow can provide useful information, but for unsteady flows it is not necessarily information about the streamlines.

EXAMPLE 4.2

Determine the streamlines for the two-dimensional steady flow discussed in Example 4.1, $\mathbf{V} = (V_0/\ell)(x\hat{\mathbf{i}} - y\hat{\mathbf{j}})$.

SOLUTION

Since $u = (V_0/\ell)x$ and $\mathrm{V} = -(V_0/\ell)y$ it follows that streamlines are given by solution of the equation

$$\frac{dy}{dx} = \frac{\mathrm{V}}{u} = \frac{-(V_0/\ell)y}{(V_0/\ell)x} = -\frac{y}{x}$$

in which variables can be separated and the equation integrated to give

$$\int \frac{dy}{y} = -\int \frac{dx}{x}$$

or

$$\ln y = -\ln x + \text{constant}$$

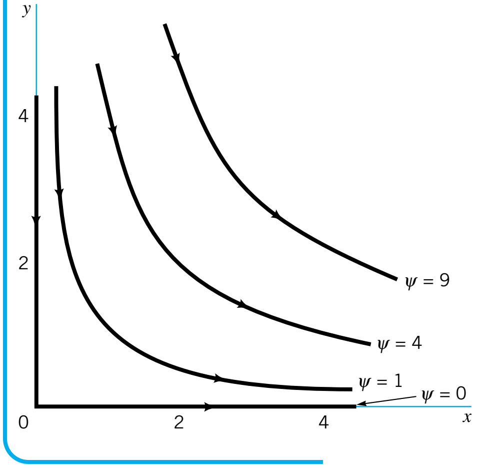
Thus, along the streamline

$$xy = C, \qquad \text{where } C \text{ is a constant} \qquad \textbf{(Ans)}$$

By using different values of the constant C, we can plot various lines in the x–y plane—the streamlines. The usual notation for a streamline is ψ = constant on a streamline. Thus, the equation for the streamlines of this flow are

$$\psi = xy$$

As is discussed more fully in Chapter 6, the function $\psi = \psi\,(x, y)$ is called the *stream function.* The streamlines in the first quadrant are plotted in Fig. E4.2. A comparison of this figure with Fig. E4.1*a* illustrates the fact that streamlines are lines parallel to the velocity field.



■ FIGURE E4.2

A *streakline* consists of all particles in a flow that have previously passed through a common point. Streaklines are more of a laboratory tool than an analytical tool. They can be obtained by taking instantaneous photographs of marked particles that all passed through a given location in the flow field at some earlier time. Such a line can be produced by continuously injecting marked fluid (neutrally buoyant smoke in air, or dye in water) at a given location (Ref. 2). (See **Fig. 9.1**.) If the flow is steady, each successively injected particle follows precisely behind the previous one, forming a steady streakline that is exactly the same as the streamline through the injection point.

V4.5 Streamlines

For unsteady flows, particles injected at the same point at different times need not follow the same path. An instantaneous photograph of the marked fluid would show the streakline at that instant, but it would not necessarily coincide with the streamline through the point of injection at that particular time nor with the streamline through the same injection point at a different time (see Example 4.3).

The third method used for visualizing and describing flows involves the use of *pathlines*. A pathline is the line traced out by a given particle as it flows from one point to another. The pathline is a Lagrangian concept that can be produced in the laboratory by marking a fluid particle (dying a small fluid element) and taking a time exposure photograph of its motion. (See the photographs at the beginning of **Chapters 5**, **7**, and **10**.)

For steady flow, streamlines, streaklines, and pathlines are the same.

If the flow is steady, the path taken by a marked particle (a pathline) will be the same as the line formed by all other particles that previously passed through the point of injection (a streakline). For such cases these lines are tangent to the velocity field. Hence, pathlines, streamlines, and streaklines are the same for steady flows. For unsteady flows none of these three types of lines need be the same (Ref. 3). Often one sees pictures of "streamlines" made visible by the injection of smoke or dye into a flow as is shown in **Fig. 4.3**. Actually, such pictures show streaklines rather than streamlines. However, for steady flows the two are identical; only the nomenclature is incorrectly used.

EXAMPLE 4.3

Water flowing from the oscillating slit shown in Fig. E4.3*a* produces a velocity field given by $\mathbf{V} = u_0 \sin[\omega(t - y/v_0)]\hat{\mathbf{i}} + v_0\hat{\mathbf{j}}$, where u_0, v_0, and ω are constants. Thus, the y component of velocity remains constant ($v = v_0$) and the x component of velocity at $y = 0$ coincides with the velocity of the oscillating sprinkler head $[u = u_0 \sin(\omega t)$ at $y = 0]$.

(a) Determine the streamline that passes through the origin at $t = 0$; at $t = \pi/2\omega$. (b) Determine the pathline of the particle that was at the origin at $t = 0$; at $t = \pi/2$. (c) Discuss the shape of the streakline that passes through the origin.

SOLUTION

(a) Since $u = u_0 \sin[\omega(t - y/v_0)]$ and $v = v_0$ it follows from Eq. 4.1 that streamlines are given by the solution of

$$\frac{dy}{dx} = \frac{v}{u} = \frac{v_0}{u_0 \sin[\omega(t - y/v_0)]}$$

in which the variables can be separated and the equation integrated (for any given time t) to give

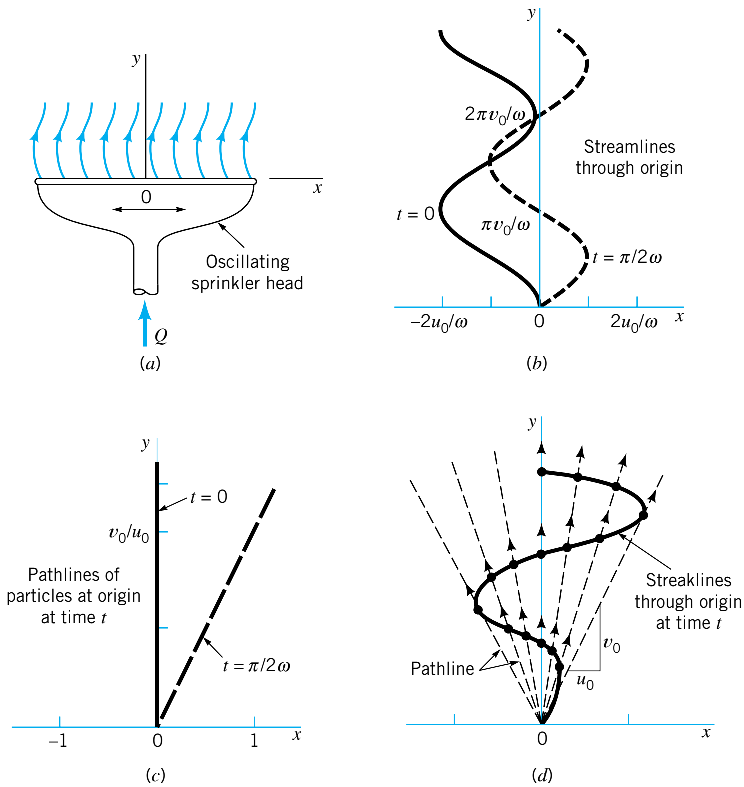
$$u_0 \int \sin\left[\omega\left(t - \frac{y}{v_0}\right)\right] dy = v_0 \int dx,$$

or

$$u_0(v_0/\omega)\cos\left[\omega\left(t - \frac{y}{v_0}\right)\right] = v_0 x + C \tag{1}$$

where C is a constant. For the streamline at $t = 0$ that passes through the origin ($x = y = 0$), the value of C is obtained from Eq. 1 as $C = u_0 v_0/\omega$. Hence, the equation for this streamline is

$$x = \frac{u_0}{\omega}\left[\cos\left(\frac{\omega y}{v_0}\right) - 1\right] \qquad \textbf{(2)} \quad \textbf{(Ans)}$$



■ FIGURE E4.3

Similarly, for the streamline at $t = \pi/2\omega$ that passes through the origin, Eq. 1 gives $C = 0$. Thus, the equation for this streamline is

$$x = \frac{u_0}{\omega}\cos\left[\omega\left(\frac{\pi}{2\omega} - \frac{y}{v_0}\right)\right] = \frac{u_0}{\omega}\cos\left(\frac{\pi}{2} - \frac{\omega y}{v_0}\right)$$

or

$$x = \frac{u_0}{\omega}\sin\left(\frac{\omega y}{v_0}\right) \qquad \textbf{(3)} \quad \textbf{(Ans)}$$

These two streamlines, plotted in Fig. E4.3b, are not the same because the flow is unsteady. For example, at the origin ($x = y = 0$) the velocity is $\mathbf{V} = v_0\,\hat{\mathbf{j}}$ at $t = 0$ and $\mathbf{V} = u_0\hat{\mathbf{i}} + v_0\,\hat{\mathbf{j}}$ at $t = \pi/2\omega$. Thus, the angle of the streamline passing through the origin changes with time. Similarly, the shape of the entire streamline is a function of time.

(b) The pathline of a particle (the location of the particle as a function of time) can be obtained from the velocity field and the definition of the velocity. Since $u = dx/dt$ and $v = dy/dt$ we obtain

$$\frac{dx}{dt} = u_0 \sin\left[\omega\left(t - \frac{y}{v_0}\right)\right] \quad \text{and} \quad \frac{dy}{dt} = v_0$$

The y equation can be integrated (since v_0 = constant) to give the y coordinate of the pathline as

$$y = v_0 t + C_1 \tag{4}$$

where C_1 is a constant. With this known $y = y(t)$ dependence, the x equation for the pathline becomes

$$\frac{dx}{dt} = u_0 \sin\left[\omega\left(t - \frac{v_0 t + C_1}{v_0}\right)\right] = -u_0 \sin\left(\frac{C_1 \omega}{v_0}\right)$$

This can be integrated to give the x component of the pathline as

$$x = -\left[u_0 \sin\left(\frac{C_1\omega}{v_0}\right)\right]t + C_2 \tag{5}$$

where C_2 is a constant. For the particle that was at the origin ($x = y = 0$) at time $t = 0$, Eqs. 4 and 5 give $C_1 = C_2 = 0$. Thus, the pathline is

$$x = 0 \quad \text{and} \quad y = v_0 t \tag{6}$$ **(Ans)**

Similarly, for the particle that was at the origin at $t = \pi/2\omega$, Eqs. 4 and 5 give $C_1 = -\pi v_0/2\omega$ and $C_2 = -\pi u_0/2\omega$. Thus, the pathline for this particle is

$$x = u_0\left(t - \frac{\pi}{2\omega}\right) \quad \text{and} \quad y = v_0\left(t - \frac{\pi}{2\omega}\right) \tag{7}$$

The pathline can be drawn by plotting the locus of $x(t)$, $y(t)$ values for $t \geq 0$ or by eliminating the parameter t from Eq. 7 to give

$$y = \frac{v_0}{u_0}x \tag{8}$$ **(Ans)**

The pathlines given by Eqs. 6 and 8, shown in Fig. E4.3*c*, are straight lines from the origin (rays). The pathlines and streamlines do not coincide because the flow is unsteady.

V4.6 Pathlines

(c) The streakline through the origin at time $t = 0$ is the locus of particles at $t = 0$ that previously ($t < 0$) passed through the origin. The general shape of the streaklines can be seen as follows. Each particle that flows through the origin travels in a straight line (pathlines are rays from the origin), the slope of which lies between $\pm\, v_0/u_0$ as shown in Fig. E4.3*d*. Particles passing through the origin at different times are located on different rays from the origin and at different distances from the origin. The net result is that a stream of dye continually injected at the origin (a streakline) would have the shape shown in Fig. E4.3*d*. Because of the unsteadiness, the streakline will vary with time, although it will always have the oscillating, sinuous character shown. Similar streaklines are given by the stream of water from a garden hose nozzle that oscillates back and forth in a direction normal to the axis of the nozzle.

In this example neither the streamlines, pathlines, nor streaklines coincide. If the flow were steady all of these lines would be the same.

4.2 The Acceleration Field

As indicated in the previous section, we can describe fluid motion by either (1) following individual particles (Lagrangian description) or (2) remaining fixed in space and observing different particles as they pass by (Eulerian description). In either case, to apply Newton's second law ($\mathbf{F} = m\mathbf{a}$) we must be able to describe the particle acceleration in an appropriate fashion. For the infrequently used Lagrangian method, we describe the fluid acceleration just as is done in solid body dynamics—$\mathbf{a} = \mathbf{a}(t)$ for each particle. For the Eulerian description we describe the *acceleration field* as a function of position and time without actually following any particular particle. This is analogous to describing the flow in terms of the velocity field, $\mathbf{V} = \mathbf{V}(x, y, z, t)$, rather than the velocity for particular particles. In this section we will discuss how to obtain the acceleration field if the velocity field is known.

The acceleration of a particle is the time rate of change of its velocity. For unsteady flows the velocity at a given point in space (occupied by different particles) may vary with time, giving rise to a portion of the fluid acceleration. In addition, a fluid particle may experience an acceleration because its velocity changes as it flows from one point to another in space. For example, water flowing through a garden hose nozzle under steady conditions (constant number of gallons per minute from the hose) will experience an acceleration as it changes from its relatively low velocity in the hose to its relatively high velocity at the tip of the nozzle.

4.2.1 The Material Derivative

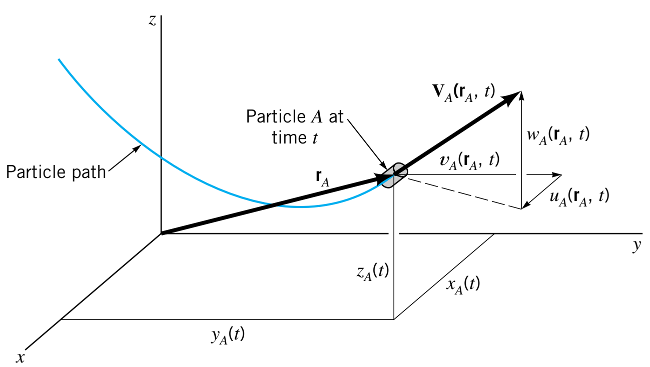
Consider a fluid particle moving along its pathline as is shown in Fig. 4.4. In general, the particle's velocity, denoted $\mathbf{V}_A$ for particle A, is a function of its location and the time. That is,

$$\mathbf{V}_A = \mathbf{V}_A(\mathbf{r}_A, t) = \mathbf{V}_A[x_A(t), y_A(t), z_A(t), t]$$

where $x_A = x_A(t)$, $y_A = y_A(t)$, and $z_A = z_A(t)$ define the location of the moving particle. By definition, the acceleration of a particle is the time rate of change of its velocity. Since the velocity may be a function of both position and time, its value may change because of the change in time as well as a change in the particle's position. Thus, we use the chain rule of differentiation to obtain the acceleration of particle A, denoted $\mathbf{a}_A$, as

Acceleration is the time rate of change of velocity for a given particle.

$$\mathbf{a}_A(t) = \frac{d\mathbf{V}_A}{dt} = \frac{\partial \mathbf{V}_A}{\partial t} + \frac{\partial \mathbf{V}_A}{\partial x}\frac{dx_A}{dt} + \frac{\partial \mathbf{V}_A}{\partial y}\frac{dy_A}{dt} + \frac{\partial \mathbf{V}_A}{\partial z}\frac{dz_A}{dt} \tag{4.2}$$



■ **FIGURE 4.4 Velocity and position of particle A at time t.**

Using the fact that the particle velocity components are given by $u_A = dx_A/dt$, $v_A = dy_A/dt$, and $w_A = dz_A/dt$, Eq. 4.2 becomes

$$\mathbf{a}_A = \frac{\partial \mathbf{V}_A}{\partial t} + u_A \frac{\partial \mathbf{V}_A}{\partial x} + v_A \frac{\partial \mathbf{V}_A}{\partial y} + w_A \frac{\partial \mathbf{V}_A}{\partial z}$$

Since the above is valid for any particle, we can drop the reference to particle A and obtain the acceleration field from the velocity field as

$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + u \frac{\partial \mathbf{V}}{\partial x} + v \frac{\partial \mathbf{V}}{\partial y} + w \frac{\partial \mathbf{V}}{\partial z} \tag{4.3}$$

This is a vector result whose scalar components can be written as

$$a_x = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \tag{4.4}$$

and

$$a_z = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z}$$

where a_x, a_y, and a_z are the x, y, and z components of the acceleration.

The above result is often written in shorthand notation as

$$\mathbf{a} = \frac{D\mathbf{V}}{Dt}$$

where the operator

$$\frac{D(\ )}{Dt} \equiv \frac{\partial(\ )}{\partial t} + u \frac{\partial(\ )}{\partial x} + v \frac{\partial(\ )}{\partial y} + w \frac{\partial(\ )}{\partial z} \tag{4.5}$$

The material derivative is used to describe time rates of change for a given particle.

is termed the *material derivative* or *substantial derivative*. An often-used shorthand notation for the material derivative operator is

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + (\mathbf{V} \cdot \nabla)(\ ) \tag{4.6}$$

The dot product of the velocity vector, $\mathbf{V}$, and the gradient operator, $\nabla(\ ) = \partial(\ )/\partial x\,\hat{\mathbf{i}} + \partial(\ )/\partial y\,\hat{\mathbf{j}} + \partial(\ )/\partial z\,\hat{\mathbf{k}}$ (a vector operator) provides a convenient notation for the spatial derivative terms appearing in the Cartesian coordinate representation of the material derivative. Note that the notation $\mathbf{V} \cdot \nabla$ represents the operator $\mathbf{V} \cdot \nabla(\ ) = u\partial(\ )/\partial x + v\partial(\ )/\partial y + w\partial(\ )/\partial z$.

The material derivative concept is very useful in analysis involving various fluid parameters, not just the acceleration. The material derivative of any variable is the rate at which that variable changes with time for a given particle (as seen by one moving along with the fluid—the Lagrangian description). For example, consider a temperature field $T = T(x, y, z, t)$ associated with a given flow, like that shown in Fig. 4.2. It may be of interest to determine the time rate of change of temperature of a fluid particle (particle A) as it moves through this temperature field. If the velocity, $\mathbf{V} = \mathbf{V}(x, y, z, t)$, is known, we can apply the chain rule to determine the rate of change of temperature as

$$\frac{dT_A}{dt} = \frac{\partial T_A}{\partial t} + \frac{\partial T_A}{\partial x}\frac{dx_A}{dt} + \frac{\partial T_A}{\partial y}\frac{dy_A}{dt} + \frac{\partial T_A}{\partial z}\frac{dz_A}{dt}$$

This can be written as

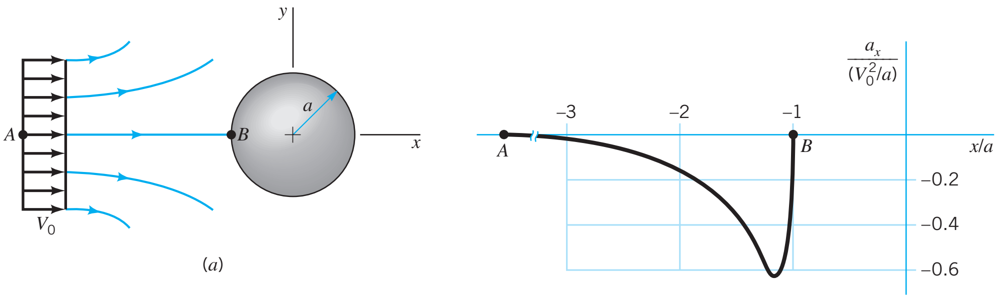
$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T$$

As in the determination of the acceleration, the material derivative operator, $D(\ )/Dt$, appears.

EXAMPLE 4.4

An incompressible, inviscid fluid flows steadily past a sphere of radius a, as shown in Fig. E4.4*a*. According to a more advanced analysis of the flow, the fluid velocity along streamline A–B is given by

$$\mathbf{V} = u(x)\hat{\mathbf{i}} = V_0\left(1 + \frac{a^3}{x^3}\right)\hat{\mathbf{i}}$$



■ FIGURE E4.4 (b)

where V_0 is the upstream velocity far ahead of the sphere. Determine the acceleration experienced by fluid particles as they flow along this streamline.

SOLUTION

Along streamline A–B there is only one component of velocity ($v = w = 0$) so that from Eq. 4.3

$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} = \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right)\hat{\mathbf{i}}$$

or

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}, \qquad a_y = 0, \qquad a_z = 0$$

Since the flow is steady the velocity at a given point in space does not change with time. Thus, $\partial u/\partial t = 0$. With the given velocity distribution along the streamline, the acceleration becomes

$$a_x = u\frac{\partial u}{\partial x} = V_0\left(1 + \frac{a^3}{x^3}\right)V_0[a^3(-3x^{-4})]$$

or

$$a_x = -3(V_0^2/a)\frac{1 + (a/x)^3}{(x/a)^4} \qquad \textbf{(Ans)}$$

Along streamline A–$B(-\infty \leq x \leq -a$ and $y = 0)$ the acceleration has only an x component and it is negative (a deceleration). Thus, the fluid slows down from its upstream velocity of $\mathbf{V} = V_0\hat{\mathbf{i}}$ at $x = -\infty$ to its stagnation point velocity of $\mathbf{V} = 0$ at $x = -a$, the "nose" of the sphere. The variation of a_x along streamline A–B is shown in Fig. E4.4*b*. It is the same result as is obtained in Example 3.1 by using the streamwise component of the acceleration, $a_x = V\,\partial V/\partial s$. The maximum deceleration occurs at $x = -1.205a$ and has a value of $a_x = -0.610V_0^2/a$.

In general, for fluid particles on streamlines other than A–B, all three components of the acceleration (a_x, a_y, and a_z) will be nonzero.

Fairly large accelerations (or decelerations) often occur in fluid flows. Consider air flowing past a baseball of radius $a = 0.14$ ft with a velocity of $V_0 = 100$ mi/hr $= 147$ ft/s. According to the results of Example 4.4, the maximum deceleration of an air particle approaching the stagnation point along the streamline in front of the ball is

$$|a_x|_{\max} = |a_x|_{x=-0.168\text{ ft}} = \frac{0.610(147\text{ ft/s})^2}{0.14\text{ ft}} = 94.2 \times 10^3\text{ ft/s}^2$$

This is a deceleration of approximately 3000 times that of gravity. In some situations the acceleration or deceleration experienced by fluid particles may be very large. An extreme case involves flow through shock waves that can occur in supersonic flow past objects (see **Chapter 11**). In such circumstances the fluid particles may experience decelerations hundreds of thousands of times greater than gravity. Large forces are obviously needed to produce such accelerations.

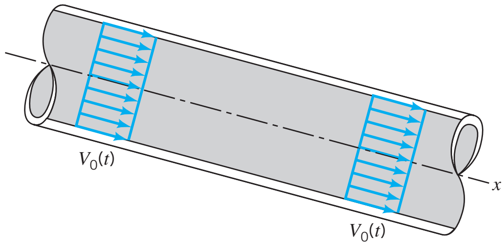
4.2.2 Unsteady Effects

The local derivative is a result of the unsteadiness of the flow.

As is seen from Eq. 4.5, the material derivative formula contains two types of terms—those involving the time derivative $[\partial(\ )/\partial t]$ and those involving spatial derivatives $[\partial(\ )/\partial x$, $\partial(\ )/\partial y$, and $\partial(\ )/\partial z]$. The time derivative portions are denoted as the *local derivative*. They represent effects of the unsteadiness of the flow. If the parameter involved is the acceleration, that portion given by $\partial\mathbf{V}/\partial t$ is termed the *local acceleration*. For steady flow the time derivative is zero throughout the flow field $[\partial(\ )/\partial t \equiv 0]$, and the local effect vanishes. Physically, there is no change in flow parameters at a fixed point in space if the flow is steady. There may be a change of those parameters for a fluid particle as it moves about, however.

If a flow is unsteady, its parameter values (velocity, temperature, density, etc.) at any location may change with time. For example, an unstirred ($\mathbf{V} = 0$) cup of coffee will cool down in time because of heat transfer to its surroundings. That is, $DT/Dt = \partial T/\partial t + \mathbf{V}\cdot\nabla T = \partial T/\partial t < 0$. Similarly, a fluid particle may have nonzero acceleration as a result of the unsteady effect of the flow. Consider flow in a constant diameter pipe as is shown in Fig. 4.5. The flow is assumed to be spatially uniform throughout the pipe. That is, $\mathbf{V} = V_0(t)\,\hat{\mathbf{i}}$ at all points in the pipe. The value of the acceleration depends on whether V_0 is being increased, $\partial V_0/\partial t > 0$, or decreased, $\partial V_0/\partial t < 0$. Unless V_0 is independent of time ($V_0 \equiv$ constant) there will be an acceleration, the local acceleration term. Thus, the acceleration field, $\mathbf{a} = \partial V_0/\partial t\,\hat{\mathbf{i}}$, is uniform throughout the entire flow, although it may vary with time ($\partial V_0/\partial t$ need not be constant). The acceleration due to the spatial variations of velocity ($u\,\partial u/\partial x$, $v\,\partial v/\partial y$, etc.) vanishes automatically for this flow, since $\partial u/\partial x = 0$ and $v = w = 0$. That is,

$$\mathbf{a} = \frac{\partial\mathbf{V}}{\partial t} + u\frac{\partial\mathbf{V}}{\partial x} + v\frac{\partial\mathbf{V}}{\partial y} + w\frac{\partial\mathbf{V}}{\partial z} = \frac{\partial\mathbf{V}}{\partial t} = \frac{\partial V_0}{\partial t}\hat{\mathbf{i}}$$



■ **FIGURE 4.5 Uniform, unsteady flow in a constant diameter pipe.**

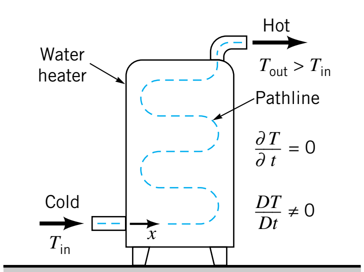
4.2.3 Convective Effects

The convective derivative is a result of the spatial variation of the flow.

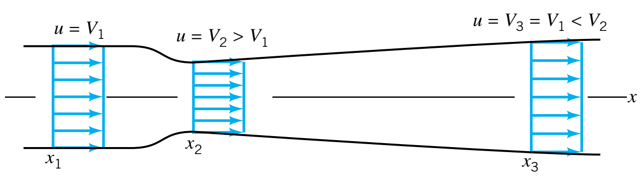
The portion of the material derivative (Eq. 4.5) represented by the spatial derivatives is termed the *convective derivative*. It represents the fact that a flow property associated with a fluid particle may vary because of the motion of the particle from one point in space where the parameter has one value to another point in space where its value is different. This contribution to the time rate of change of the parameter for the particle can occur whether the flow is steady or unsteady. It is due to the convection, or motion, of the particle through space in which there is a gradient $[\nabla(\ ) = \partial(\ )/\partial x\,\hat{\mathbf{i}} + \partial(\ )/\partial y\,\hat{\mathbf{j}} + \partial(\ )/\partial z\,\hat{\mathbf{k}}]$ in the parameter value. That portion of the acceleration given by the term $(\mathbf{V} \cdot \nabla)\mathbf{V}$ is termed the *convective acceleration.*

As is illustrated in Fig. 4.6, the temperature of a water particle changes as it flows through a water heater. The water entering the heater is always the same cold temperature and the water leaving the heater is always the same hot temperature. The flow is steady. However, the temperature, T, of each water particle increases as it passes through the heater—$T_{\text{out}} > T_{\text{in}}$. Thus, $DT/Dt \neq 0$ because of the convective term in the total derivative of the temperature. That is, $\partial T/\partial t = 0$, but $u\,\partial T/\partial x \neq 0$ (where x is directed along the streamline), since there is a nonzero temperature gradient along the streamline. A fluid particle traveling along this nonconstant temperature path $(\partial T/\partial x \neq 0)$ at a specified speed (u) will have its temperature change with time at a rate of $DT/Dt = u\,\partial T/\partial x$ even though the flow is steady $(\partial T/\partial t = 0)$.

The same types of processes are involved with fluid accelerations. Consider flow in a variable area pipe as shown in Fig. 4.7. It is assumed that the flow is steady and one-dimensional with velocity that increases and decreases in the flow direction as indicated. As the fluid flows from section (1) to section (2), its velocity increases from V_1 to V_2. Thus, even though $\partial\mathbf{V}/\partial t = 0$, fluid particles experience an acceleration given by $a_x = u\,\partial u/\partial x$. For $x_1 < x < x_2$, it is seen that $\partial u/\partial x > 0$ so that $a_x > 0$—the fluid accelerates. For $x_2 < x < x_3$, it is seen that $\partial u/\partial x < 0$ so that $a_x < 0$—the fluid decelerates. If $V_1 = V_3$, the amount of acceleration precisely balances the amount of deceleration even though the distances between x_2 and x_1 and x_3 and x_2 are not the same.



■ **FIGURE 4.6 Steady-state operation of a water heater.**



■ **FIGURE 4.7 Uniform, steady flow in a variable area pipe.**

EXAMPLE 4.5

Consider the steady, two-dimensional flow field discussed in Example 4.2. Determine the acceleration field for this flow.

SOLUTION

In general, the acceleration is given by

$$\mathbf{a} = \frac{D\mathbf{V}}{Dt} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)(\mathbf{V}) = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} \quad \textbf{(1)}$$

where the velocity is given by $\mathbf{V} = (V_0/\ell)(x\hat{\mathbf{i}} - y\hat{\mathbf{j}})$ so that $u = (V_0/\ell)x$ and $v = -(V_0/\ell)y$. For steady $[\partial(\ )/\partial t = 0]$, two-dimensional $[w = 0$ and $\partial(\ )/\partial z = 0]$ flow, Eq. 1 becomes

$$\mathbf{a} = u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} = \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)\hat{\mathbf{i}} + \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)\hat{\mathbf{j}}$$

Hence, for this flow the acceleration is given by

$$\mathbf{a} = \left[\left(\frac{V_0}{\ell}\right)(x)\left(\frac{V_0}{\ell}\right) + \left(\frac{V_0}{\ell}\right)(y)(0)\right]\hat{\mathbf{i}} + \left[\left(\frac{V_0}{\ell}\right)(x)(0) + \left(\frac{-V_0}{\ell}\right)(y)\left(\frac{-V_0}{\ell}\right)\right]\hat{\mathbf{j}}$$

or

$$a_x = \frac{V_0^2 x}{\ell^2}, \qquad a_y = \frac{V_0^2 y}{\ell^2} \qquad \textbf{(Ans)}$$

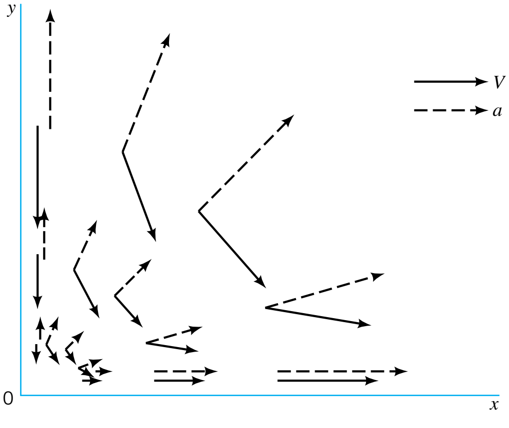
The fluid experiences an acceleration in both the x and y directions. Since the flow is steady, there is no local acceleration—the fluid velocity at any given point is constant in time. However, there is a convective acceleration due to the change in velocity from one point on the particle's pathline to another. Recall that the velocity is a vector—it has both a magnitude and a direction. In this flow both the fluid speed (magnitude) and flow direction change with location (see Fig. E4.1*a*).

For this flow the magnitude of the acceleration is constant on circles centered at the origin, as is seen from the fact that

$$|\mathbf{a}| = (a_x^2 + a_y^2 + a_z^2)^{1/2} = \left(\frac{V_0}{\ell}\right)^2 (x^2 + y^2)^{1/2} \quad \textbf{(2)}$$

Also, the acceleration vector is oriented at an angle θ from the x axis, where

$$\tan\theta = \frac{a_y}{a_x} = \frac{y}{x}$$



■ FIGURE E4.5

This is the same angle as that formed by a ray from the origin to point (x, y). Thus, the acceleration is directed along rays from the origin and has a magnitude proportional to the distance from the origin. Typical acceleration vectors (from Eq. 2) and velocity vectors (from Example 4.1) are shown in Fig. E4.5 for the flow in the first quadrant. Note that **a** and **V** are not parallel except along the x and y axes (a fact that is responsible for the curved pathlines of the flow), and that both the acceleration and velocity are zero at the origin $(x = y = 0)$. An infinitesimal fluid particle placed precisely at the origin will remain there, but its neighbors (no matter how close they are to the origin) will drift away.

The concept of the material derivative can be used to determine the time rate of change of any parameter associated with a particle as it moves about. Its use is not restricted to fluid mechanics alone. The basic ingredients needed to use the material derivative concept are the field description of the parameter, $P = P(x, y, z, t)$, and the rate at which the particle moves through that field, $\mathbf{V} = \mathbf{V}(x, y, z, t)$.

EXAMPLE 4.6

A manufacturer produces a perishable product in a factory located at $x = 0$ and sells the product along the distribution route $x > 0$. The selling price of the product, P, is a function of the length of time after it was produced, t, and the location at which it is sold, x. That is, $P = P(x, t)$. At a given location the price of the product decreases in time (it is perishable) according to $\partial P/\partial t = -C_1$, where C_1 is a positive constant (dollars per hour). In addition, because of shipping costs the price increases with distance from the factory according to $\partial P/\partial x = C_2$, where C_2 is a positive constant (dollars per mile). If the manufacturer wishes to sell the product for the same price anywhere along the distribution route, determine how fast he must travel along the route.

SOLUTION

For a given batch of the product (Lagrangian description), the time rate of change of the price can be obtained by using the material derivative

$$\frac{DP}{Dt} = \frac{\partial P}{\partial t} + \mathbf{V} \cdot \nabla P = \frac{\partial P}{\partial t} + u\frac{\partial P}{\partial x} + v\frac{\partial P}{\partial y} + w\frac{\partial P}{\partial z} = \frac{\partial P}{\partial t} + u\frac{\partial P}{\partial x}$$

We have used the fact that the motion is one-dimensional with $\mathbf{V} = u\hat{\mathbf{i}}$, where u is the speed at which the product is convected along its route. If the price is to remain constant as the product moves along the distribution route, then

$$\frac{DP}{Dt} = 0 \quad \text{or} \quad \frac{\partial P}{\partial t} + u\frac{\partial P}{\partial x} = 0$$

Thus, the correct delivery speed is

$$u = \frac{-\partial P/\partial t}{\partial P/\partial x} = \frac{C_1}{C_2} \qquad \textbf{(Ans)}$$

With this speed, the decrease in price because of the local effect $(\partial P/\partial t)$ is exactly balanced by the increase in price due to the convective effect $(u\, \partial P/\partial x)$. A faster delivery speed will cause the price of the given batch of the product to increase in time ($DP/Dt > 0$; it is rushed to distant markets before it spoils), while a slower delivery speed will cause its price to decrease ($DP/Dt < 0$; the increased costs due to distance from the factory is more than offset by reduced costs due to spoilage).

4.2.4 Streamline Coordinates

In many flow situations it is convenient to use a coordinate system defined in terms of the streamlines of the flow. An example for steady, two-dimensional flows is illustrated in Fig. 4.8. Such flows can be described either in terms of the usual x, y Cartesian coordinate system (or some other system such as the r, θ polar coordinate system) or the streamline coordinate system. In the streamline coordinate system the flow is described in terms of one coordinate along the streamlines, denoted s, and the second coordinate normal to the streamlines, denoted n. Unit vectors in these two directions are denoted by $\hat{\mathbf{s}}$ and $\hat{\mathbf{n}}$, as shown in the figure. Care is needed not to confuse the coordinate distance s (a scalar) with the unit vector along the streamline direction, $\hat{\mathbf{s}}$.

Streamline coordinates provide a natural coordinate system for a flow.

The flow plane is therefore covered by an orthogonal curved net of coordinate lines. At any point the s and n directions are perpendicular, but the lines of constant s or constant n are not necessarily straight. Without knowing the actual velocity field (hence, the streamlines) it is not possible to construct this flow net. In many situations appropriate simplifying assumptions can be made so that this lack of information does not present an insurmountable difficulty. One of the major advantages of using the streamline coordinate system is that the velocity is always tangent to the s direction. That is,

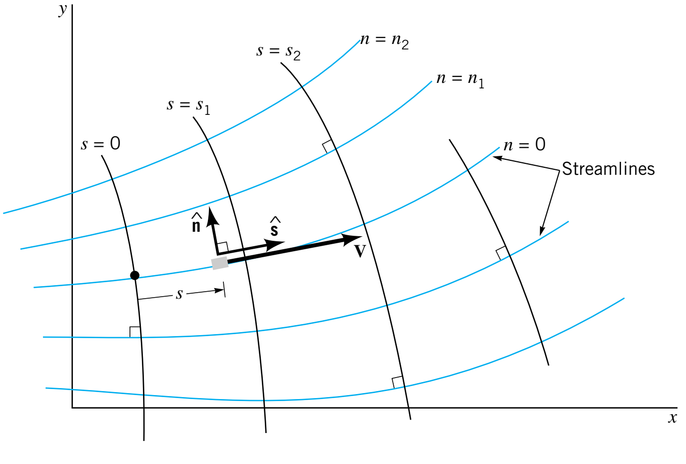
$$\mathbf{V} = V\hat{\mathbf{s}}$$

This allows simplifications in describing the fluid particle acceleration and in solving the equations governing the flow.

For steady, two-dimensional flow we can determine the acceleration as

$$\mathbf{a} = \frac{D\mathbf{V}}{Dt} = a_s\hat{\mathbf{s}} + a_n\hat{\mathbf{n}}$$

where a_s and a_n are the streamline and normal components of acceleration, respectively. We use the material derivative because by definition the acceleration is the time rate of change of the velocity of a given particle as it moves about. If the streamlines are curved, both the speed of the particle and its direction of flow may change from one point to another. In general, for steady flow both the speed and the flow direction are a function of location—$V = V(s, n)$ and $\hat{\mathbf{s}} = \hat{\mathbf{s}}(s, n)$. For a given particle, the value of s changes with time, but the value of n remains fixed because the particle flows along a streamline defined by n = constant. (Recall that streamlines and pathlines coincide in steady flow.) Thus, application of the chain rule gives



■ FIGURE 4.8 **Streamline coordinate system for two-dimensional flow.**

$$\mathbf{a} = \frac{D(V\,\hat{\mathbf{s}})}{Dt} = \frac{DV}{Dt}\hat{\mathbf{s}} + V\frac{D\hat{\mathbf{s}}}{Dt}$$

or

$$\mathbf{a} = \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial s}\frac{ds}{dt} + \frac{\partial V}{\partial n}\frac{dn}{dt}\right)\hat{\mathbf{s}} + V\left(\frac{\partial \hat{\mathbf{s}}}{\partial t} + \frac{\partial \hat{\mathbf{s}}}{\partial s}\frac{ds}{dt} + \frac{\partial \hat{\mathbf{s}}}{\partial n}\frac{dn}{dt}\right)$$

This can be simplified by using the fact that for steady flow nothing changes with time at a given point so that both $\partial V/\partial t$ and $\partial \hat{\mathbf{s}}/\partial t$ are zero. Also, the velocity along the streamline is $V = ds/dt$ and the particle remains on its streamline ($n =$ constant) so that $dn/dt = 0$. Hence,

$$\mathbf{a} = \left(V\frac{\partial V}{\partial s}\right)\hat{\mathbf{s}} + V\left(V\frac{\partial \hat{\mathbf{s}}}{\partial s}\right)$$

The quantity $\partial \hat{\mathbf{s}}/\partial s$ represents the limit as $\delta s \rightarrow 0$ of the change in the unit vector along the streamline, $\delta \hat{\mathbf{s}}$, per change in distance along the streamline, δs. The magnitude of $\hat{\mathbf{s}}$ is constant ($|\hat{\mathbf{s}}| = 1$; it is a unit vector), but its direction is variable if the streamlines are curved. From Fig. 4.9 it is seen that the magnitude of $\partial \hat{\mathbf{s}}/\partial s$ is equal to the inverse of the radius of curvature of the streamline, $\mathcal{R}$, at the point in question. This follows because the two triangles shown (AOB and $A'O'B'$) are similar triangles so that $\delta s/\mathcal{R} = |\delta \hat{\mathbf{s}}|/|\hat{\mathbf{s}}| = |\delta \hat{\mathbf{s}}|$, or $|\delta \hat{\mathbf{s}}/\delta s| = 1/\mathcal{R}$. Similarly, in the limit $\delta s \rightarrow 0$, the direction of $\delta \hat{\mathbf{s}}/\delta s$ is seen to be normal to the streamline. That is,

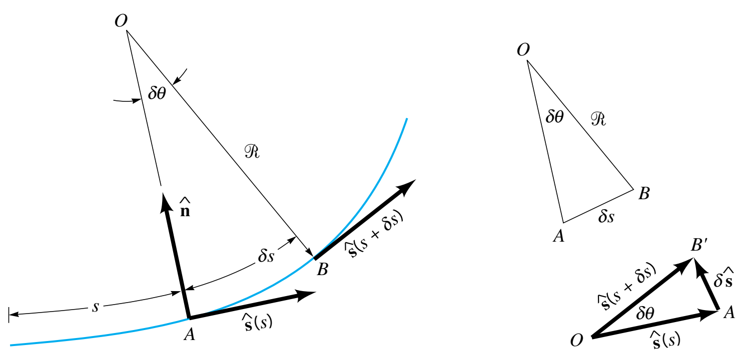
$$\frac{\partial \hat{\mathbf{s}}}{\partial s} = \lim_{\delta s \to 0} \frac{\delta \hat{\mathbf{s}}}{\delta s} = \frac{\hat{\mathbf{n}}}{\mathcal{R}}$$

Streamline and normal components of acceleration occur even in steady flows.

Hence, the acceleration for steady, two-dimensional flow can be written in terms of its streamwise and normal components in the form

$$\mathbf{a} = V\frac{\partial V}{\partial s}\hat{\mathbf{s}} + \frac{V^2}{\mathcal{R}}\hat{\mathbf{n}} \quad \text{or} \quad a_s = V\frac{\partial V}{\partial s}, \qquad a_n = \frac{V^2}{\mathcal{R}} \tag{4.7}$$

The first term, $a_s = V\,\partial V/\partial s$, represents the convective acceleration along the streamline and the second term, $a_n = V^2/\mathcal{R}$, represents centrifugal acceleration (one type of convective acceleration) normal to the fluid motion. These components can be noted in Fig. E4.5 by resolving the acceleration vector into its components along and normal to the velocity vector. Note that the unit vector $\hat{\mathbf{n}}$ is directed from the streamline toward the center of curvature. These forms of the acceleration are probably familiar from previous dynamics or physics considerations.



■ **FIGURE 4.9** **Relationship between the unit vector along the streamline, $\hat{\mathbf{s}}$, and the radius of curvature of the streamline, $\mathcal{R}$.**

4.3 Control Volume and System Representations

As is discussed in **Chapter 1**, a fluid is a type of matter that is relatively free to move and interact with its surroundings. As with any matter, a fluid's behavior is governed by a set of fundamental physical laws which are approximated by an appropriate set of equations. The application of laws such as the conservation of mass, Newton's laws of motion, and the laws of thermodynamics form the foundation of fluid mechanics analyses. There are various ways that these governing laws can be applied to a fluid, including the system approach and the control volume approach. By definition, a *system* is a collection of matter of fixed identity (always the same atoms or fluid particles), which may move, flow, and interact with its surroundings. A *control volume,* on the other hand, is a volume in space (a geometric entity, independent of mass) through which fluid may flow.

Both control volume and system concepts can be used to describe fluid flow.

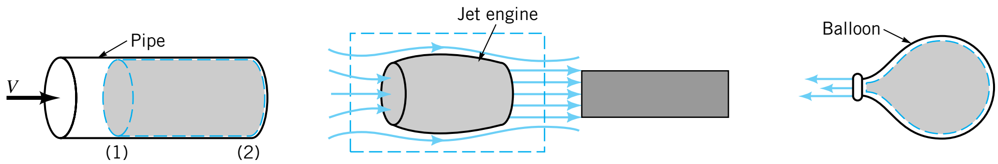
A system is a specific, identifiable quantity of matter. It may consist of a relatively large amount of mass (such as all of the air in the earth's atmosphere), or it may be an infinitesimal size (such as a single fluid particle). In any case, the molecules making up the system are "tagged" in some fashion (dyed red, either actually or only in your mind) so that they can be continually identified as they move about. The system may interact with its surroundings by various means (by the transfer of heat or the exertion of a pressure force, for example). It may continually change size and shape, but it always contains the same mass.

A mass of air drawn into an air compressor can be considered as a system. It changes shape and size (it is compressed), its temperature may change, and it is eventually expelled through the outlet of the compressor. The matter associated with the original air drawn into the compressor remains as a system, however. The behavior of this material could be investigated by applying the appropriate governing equations to this system.

One of the important concepts used in the study of statics and dynamics is that of the free-body diagram. That is, we identify an object, isolate it from its surroundings, replace its surroundings by the equivalent actions that they put on the object, and apply Newton's laws of motion. The body in such cases is our system—an identified portion of matter that we follow during its interactions with its surroundings. In fluid mechanics, it is often quite difficult to identify and keep track of a specific quantity of matter. A finite portion of a fluid contains an uncountable number of fluid particles that move about quite freely, unlike a solid that may deform but usually remains relatively easy to identify. For example, we cannot as easily follow a specific portion of water flowing in a river as we can follow a branch floating on its surface.

We may often be more interested in determining the forces put on a fan, airplane, or automobile by air flowing past the object than we are in the information obtained by following a given portion of the air (a system) as it flows along. For these situations we often use the control volume approach. We identify a specific volume in space (a volume associated with the fan, airplane, or automobile, for example) and analyze the fluid flow within, through, or around that volume. In general, the control volume can be a moving volume, although for most situations considered in this book we will use only fixed, nondeformable control volumes. The matter within a control volume may change with time as the fluid flows through it. Similarly, the amount of mass within the volume may change with time. The control volume itself is a specific geometric entity, independent of the flowing fluid.

Examples of control volumes and *control surfaces* (the surface of the control volume) are shown in Fig. 4.10. For case (*a*), fluid flows through a pipe. The fixed control surface consists of the inside surface of the pipe, the outlet end at section (2), and a section across the pipe at (1). One portion of the control surface is a physical surface (the pipe), while the remainder is simply a surface in space (across the pipe). Fluid flows across part of the control surface, but not across all of it.



■ **FIGURE 4.10** **Typical control volumes: (*a*) fixed control volume, (*b*) fixed or moving control volume, (*c*) deforming control volume.**

Another control volume is the rectangular volume surrounding the jet engine shown in Fig. 4.10*b*. If the airplane to which the engine is attached is sitting still on the runway, air flows through this control volume because of the action of the engine within it. The air that was within the engine itself at time $t = t_1$ (a system) has passed through the engine and is outside of the control volume at a later time $t = t_2$ as indicated. At this later time other air (a different system) is within the engine. If the airplane is moving, the control volume is fixed relative to an observer on the airplane, but it is a moving control volume relative to an observer on the ground. In either situation air flows through and around the engine as indicated.

The deflating balloon shown in Fig. 4.10*c* provides an example of a deforming control volume. As time increases, the control volume (whose surface is the inner surface of the balloon) decreases in size. If we do not hold onto the balloon, it becomes a moving, deforming control volume as it darts about the room. The majority of the problems we will analyze can be solved by using a fixed, nondeforming control volume. In some instances, however, it will be advantageous, in fact necessary, to use a moving, deforming control volume.

In many ways the relationship between a system and a control volume is similar to the relationship between the Lagrangian and Eulerian flow description introduced in Section 4.1.1. In the system or Lagrangian description, we follow the fluid and observe its behavior as it moves about. In the control volume or Eulerian description we remain stationary and observe the fluid's behavior at a fixed location. (If a moving control volume is used, it virtually never moves with the system—the system flows through the control volume.) These ideas are discussed in more detail in the next section.

The governing laws of fluid motion are stated in terms of fluid systems, not control volumes.

All of the laws governing the motion of a fluid are stated in their basic form in terms of a system approach. For example, "the mass of a system remains constant," or "the time rate of change of momentum of a system is equal to the sum of all the forces acting on the system." Note the word system, not control volume, in these statements. To use the governing equations in a control volume approach to problem solving, we must rephrase the laws in an appropriate manner. To this end we introduce the Reynolds transport theorem in the following section.

4.4 The Reynolds Transport Theorem

We are sometimes interested in what happens to a particular part of the fluid as it moves about. Other times we may be interested in what effect the fluid has on a particular object or volume in space as fluid interacts with it. Thus, we need to describe the laws governing fluid motion using both system concepts (consider a given mass of the fluid) and control volume concepts (consider a given volume). To do this we need an analytical tool to shift from one representation to the other. The *Reynolds transport theorem* provides this tool.

All physical laws are stated in terms of various physical parameters. Velocity, acceleration, mass, temperature, and momentum are but a few of the more common parameters. Let B represent any of these (or other) fluid parameters and b represent the amount of that parameter per unit mass. That is,

$$B = mb$$

where m is the mass of the portion of fluid of interest. For example, if $B = m$, the mass, it follows that $b = 1$. (The mass per unit mass is unity.) If $B = mV^2/2$, the kinetic energy of the mass, then $b = V^2/2$, the kinetic energy per unit mass. The parameters B and b may be scalars or vectors. Thus, if $\mathbf{B} = m\mathbf{V}$, the momentum of the mass, then $\mathbf{b} = \mathbf{V}$. (The momentum per unit mass is the velocity.)

The parameter B is termed an *extensive property* and the parameter b is termed an *intensive property*. The value of B is directly proportional to the amount of the mass being considered, whereas the value of b is independent of the amount of mass. The amount of an extensive property that a system possesses at a given instant, B_{sys}, can be determined by adding up the amount associated with each fluid particle in the system. For infinitesimal fluid particles of size $\delta V\!\!\!\!-$ and mass $\rho\, \delta V\!\!\!\!-$, this summation (in the limit of $\delta V\!\!\!\!- \rightarrow 0$) takes the form of an integration over all the particles in the system and can be written as

$$B_{\text{sys}} = \lim_{\delta V\!\!\!\!- \rightarrow 0} \sum_i b_i (\rho_i\, \delta V\!\!\!\!-_i) = \int_{\text{sys}} \rho b\, dV\!\!\!\!-$$

The limits of integration cover the entire system—a (usually) moving volume. We have used the fact that the amount of B in a fluid particle of mass $\rho\, \delta V\!\!\!\!-$ is given in terms of b by $\delta B = b\rho\, \delta V\!\!\!\!-$.

Most of the laws governing fluid motion involve the time rate of change of an extensive property of a fluid system—the rate at which the momentum of a system changes with time, the rate at which the mass of a system changes with time, and so on. Thus, we often encounter terms such as

$$\frac{dB_{\text{sys}}}{dt} = \frac{d\left(\int_{\text{sys}} \rho b\, dV\!\!\!\!-\right)}{dt} \tag{4.8}$$

To formulate the laws into a control volume approach, we must obtain an expression for the time rate of change of an extensive property within a control volume, B_{cv}, not within a system. This can be written as

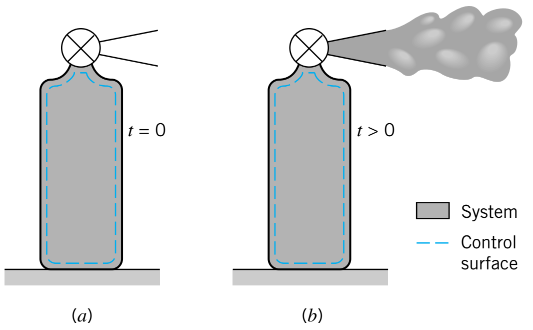
$$\frac{dB_{\text{cv}}}{dt} = \frac{d\left(\int_{\text{cv}} \rho b\, dV\!\!\!\!-\right)}{dt} \tag{4.9}$$

Differences between control volume and system concepts are subtle but very important.

where the limits of integration, denoted by cv, cover the control volume of interest. Although Eqs. 4.8 and 4.9 may look very similar, the physical interpretation of each is quite different. Mathematically, the difference is represented by the difference in the limits of integration. Recall that the control volume is a volume in space (in most cases stationary, although if it moves it need not move with the system). On the other hand, the system is an identifiable collection of mass that moves with the fluid (indeed it is a specified portion of the fluid). We will learn that even for those instances when the control volume and the system momentarily occupy the same volume in space, the two quantities dB_{sys}/dt and dB_{cv}/dt need not be the same. The Reynolds transport theorem provides the relationship between the time rate of change of an extensive property for a system and that for a control volume—the relationship between Eqs. 4.8 and 4.9.

EXAMPLE 4.7

Fluid flows from the fire extinguisher tank shown in Fig. E4.7. Discuss the differences between dB_{sys}/dt and dB_{cv}/dt if B represents mass.



■ FIGURE E4.7

SOLUTION

With $B = m$, the system mass, it follows that $b = 1$ and Eqs. 4.8 and 4.9 can be written as

$$\frac{dB_{\text{sys}}}{dt} \equiv \frac{dm_{\text{sys}}}{dt} = \frac{d\left(\int_{\text{sys}} \rho \, d\forall\right)}{dt}$$

and

$$\frac{dB_{\text{cv}}}{dt} \equiv \frac{dm_{\text{cv}}}{dt} = \frac{d\left(\int_{\text{cv}} \rho \, d\forall\right)}{dt}$$

Physically these represent the time rate of change of mass within the system and the time rate of change of mass within the control volume, respectively. We choose our system to be the fluid within the tank at the time the valve was opened ($t = 0$) and the control volume to be the tank itself. A short time after the valve is opened, part of the system has moved outside of the control volume as is shown in Fig. E4.7*b*. The control volume remains fixed. The limits of integration are fixed for the control volume; they are a function of time for the system.

Clearly, if mass is to be conserved (one of the basic laws governing fluid motion), the mass of the fluid in the system is constant, so that

$$\frac{d\left(\int_{\text{sys}} \rho \, d\forall\right)}{dt} = 0$$

On the other hand, it is equally clear that some of the fluid has left the control volume through the nozzle on the tank. Hence, the amount of mass within the tank (the control volume) decreases with time, or

$$\frac{d\left(\int_{\text{cv}} \rho \, d\forall\right)}{dt} < 0$$

The actual numerical value of the rate at which the mass in the control volume decreases will depend on the rate at which the fluid flows through the nozzle (that is, the size of the nozzle and the speed and density of the fluid). Clearly the meanings of dB_{sys}/dt and dB_{cv}/dt are different. For this example, $dB_{\text{cv}}/dt < dB_{\text{sys}}/dt$. Other situations may have $dB_{\text{cv}}/dt \geq dB_{\text{sys}}/dt$.

4.4.1 Derivation of the Reynolds Transport Theorem

The moving system flows through the fixed control volume.

A simple version of the Reynolds transport theorem relating system concepts to control volume concepts can be obtained easily for the one-dimensional flow through a fixed control volume as is shown in Fig. 4.11*a*. We consider the control volume to be that stationary volume within the pipe or duct between sections (1) and (2) as indicated. The system that we consider is that fluid occupying the control volume at some initial time t. A short time later, at time $t + \delta t$, the system has moved slightly to the right. The fluid particles that coincided with section (2) of the control surface at time t have moved a distance $\delta\ell_2 = V_2\,\delta t$ to the right, where V_2 is the velocity of the fluid as it passes section (2). Similarly, the fluid initially at section (1) has moved a distance $\delta\ell_1 = V_1\,\delta t$, where V_1 is the fluid velocity at section (1). We assume the fluid flows across sections (1) and (2) in a direction normal to these surfaces and that V_1 and V_2 are constant across sections (1) and (2).

As is shown in Fig. 4.11*b*, the outflow from the control volume from time t to $t + \delta t$ is denoted as volume II, the inflow as volume I, and the control volume itself as CV. Thus, the system at time t consists of the fluid in section CV ("SYS = CV" at time t), while at time $t + \delta t$ the system consists of the same fluid that now occupies sections (CV − I) + II. That is, "SYS = CV − I + II" at time $t + \delta t$. The control volume remains as section CV for all time.

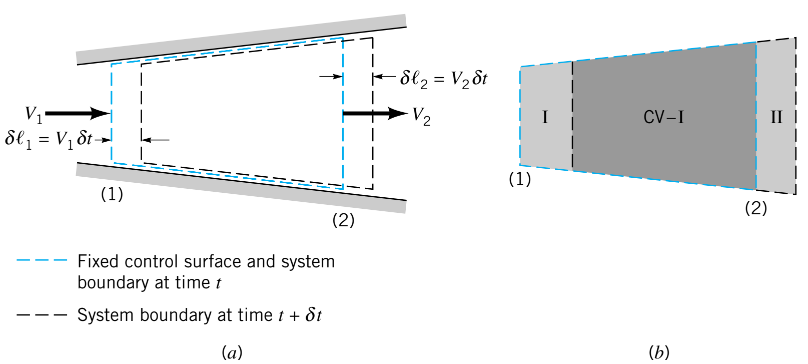
If B is an extensive parameter of the system, then the value of it for the system at time t is

$$B_{\text{sys}}(t) = B_{\text{cv}}(t)$$

since the system and the fluid within the control volume coincide at this time. Its value at time $t + \delta t$ is

$$B_{\text{sys}}(t + \delta t) = B_{\text{cv}}(t + \delta t) - B_{\text{I}}(t + \delta t) + B_{\text{II}}(t + \delta t)$$

Thus, the change in the amount of B in the system in the time interval δt divided by this time



■ **FIGURE 4.11 Control volume and system for flow through a variable area pipe.**

interval is given by

$$\frac{\delta B_{\text{sys}}}{\delta t} = \frac{B_{\text{sys}}(t + \delta t) - B_{\text{sys}}(t)}{\delta t} = \frac{B_{\text{cv}}(t + \delta t) - B_{\text{I}}(t + \delta t) + B_{\text{II}}(t + \delta t) - B_{\text{sys}}(t)}{\delta t}$$

By using the fact that at the initial time t we have $B_{\text{sys}}(t) = B_{\text{cv}}(t)$, this ungainly expression may be rearranged as follows.

$$\frac{\delta B_{\text{sys}}}{\delta t} = \frac{B_{\text{cv}}(t + \delta t) - B_{\text{cv}}(t)}{\delta t} - \frac{B_{\text{I}}(t + \delta t)}{\delta t} + \frac{B_{\text{II}}(t + \delta t)}{\delta t} \tag{4.10}$$

In the limit $\delta t \rightarrow 0$, the left-hand side of Eq. 4.10 is equal to the time rate of change of B for the system and is denoted as DB_{sys}/Dt. We use the material derivative notation, $D(\ \ )/Dt$, to denote this time rate of change to emphasize the Lagrangian character of this term. (Recall from **Section 4.2.1** that the material derivative, DP/Dt, of any quantity P represents the time rate of change of that quantity associated with a given fluid particle as it moves along.) Similarly, the quantity DB_{sys}/Dt represents the time rate of change of property B associated with a system (a given portion of fluid) as it moves along.

In the limit $\delta t \rightarrow 0$, the first term on the right-hand side of Eq. 4.10 is seen to be the time rate of change of the amount of B within the control volume

$$\lim_{\delta t \rightarrow 0} \frac{B_{\text{cv}}(t + \delta t) - B_{\text{cv}}(t)}{\delta t} = \frac{\partial B_{\text{cv}}}{\partial t} = \frac{\partial \left(\int_{\text{cv}} \rho b \, d\mathcal{V} \right)}{\partial t} \tag{4.11}$$

The third term on the right-hand side of Eq. 4.10 represents the rate at which the extensive parameter B flows from the control volume, across the control surface. This can be seen from the fact that the amount of B within region II, the outflow region, is its amount per unit volume, ρb, times the volume $\delta \mathcal{V}_{\text{II}} = A_2 \, \delta \ell_2 = A_2 \, (V_2 \delta t)$. Hence,

$$B_{\text{II}} \, (t + \delta t) = (\rho_2 b_2)(\delta \mathcal{V}_{\text{II}}) = \rho_2 b_2 A_2 V_2 \, \delta t$$

where b_2 and ρ_2 are the constant values of b and ρ across section (2). Thus, the rate at which this property flows from the control volume, $\dot{B}_{\text{out}}$, is given by

$$\dot{B}_{\text{out}} = \lim_{\delta t \rightarrow 0} \frac{B_{\text{II}}(t + \delta t)}{\delta t} = \rho_2 A_2 V_2 \, b_2 \tag{4.12}$$

Similarly, the inflow of B into the control volume across section (1) during the time interval δt corresponds to that in region I and is given by the amount per unit volume times the volume, $\delta \mathcal{V}_{\text{I}} = A_1 \, \delta \ell_1 = A_1 (V_1 \, \delta t)$. Hence,

$$B_{\text{I}} \, (t + \delta t) = (\rho_1 b_1)(\delta \mathcal{V}_1) = \rho_1 b_1 A_1 V_1 \, \delta t$$

where b_1 and ρ_1 are the constant values of b and ρ across section (1). Thus, the rate on inflow of the property B into the control volume, $\dot{B}_{\text{in}}$, is given by

$$\dot{B}_{\text{in}} = \lim_{\delta t \rightarrow 0} \frac{B_{\text{I}}(t + \delta t)}{\delta t} = \rho_1 A_1 V_1 b_1 \tag{4.13}$$

If we combine Eqs. 4.10, 4.11, 4.12, and 4.13 we see that the relationship between the time rate of change of B for the system and that for the control volume is given by

The time derivative associated with a system may be different from that for a control volume.

$$\frac{DB_{\text{sys}}}{Dt} = \frac{\partial B_{\text{cv}}}{\delta t} + \dot{B}_{\text{out}} - \dot{B}_{\text{in}} \tag{4.14}$$

or

$$\frac{DB_{\text{sys}}}{Dt} = \frac{\partial B_{\text{cv}}}{\partial t} + \rho_2 A_2 V_2 b_2 - \rho_1 A_1 V_1 b_1 \tag{4.15}$$

The Reynolds transport theorem involves time derivatives and flow rates.

This is a version of the Reynolds transport theorem valid under the restrictive assumptions associated with the flow shown in Fig. 4.11—fixed control volume with one inlet and one outlet having uniform properties (density, velocity, and the parameter b) across the inlet and outlet with the velocity normal to sections (1) and (2). Note that the time rate of change of B for the system (the left-hand side of Eq. 4.15 or the quantity in Eq. 4.8) is not necessarily the same as the rate of change of B within the control volume (the first term on the right-hand side of Eq. 4.15 or the quantity in Eq. 4.9). This is true because the inflow rate $(b_1\rho_1 V_1 A_1)$ and the outflow rate $(b_2\rho_2 V_2 A_2)$ of the property B for the control volume need not be the same.

EXAMPLE 4.8

Consider again the flow from the fire extinguisher shown in Fig. E4.7. Let the extensive property of interest be the system mass ($B = m$, the system mass, or $b = 1$) and write the appropriate form of the Reynolds transport theorem for this flow.

SOLUTION

Again we take the control volume to be the fire extinguisher, and the system to be the fluid within it at time $t = 0$. For this case there is no inlet, section (1), across which the fluid flows into the control volume ($A_1 = 0$). There is, however, an outlet, section (2). Thus, the Reynolds transport theorem, Eq. 4.15, along with Eq. 4.9 with $b = 1$ can be written as

$$\frac{Dm_{\text{sys}}}{Dt} = \frac{\partial\left(\int_{\text{cv}} \rho \, d\forall\right)}{\partial t} + \rho_2 A_2 V_2 \qquad \textbf{(1)} \quad \textbf{(Ans)}$$

If we proceed one step further and use the basic law of conservation of mass, we may set the left-hand side of this equation equal to zero (the amount of mass in a system is constant) and rewrite Eq. 1 in the form

$$\frac{\partial\left(\int_{\text{cv}} \rho \, d\forall\right)}{\partial t} = -\rho_2 A_2 V_2 \tag{2}$$

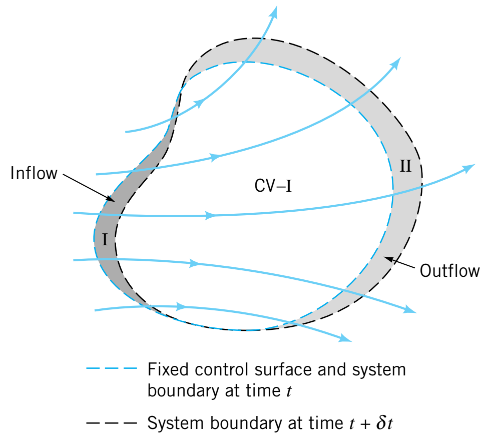
The physical interpretation of this result is that the rate at which the mass in the tank decreases in time is equal in magnitude but opposite to the rate of flow of mass from the exit, $\rho_2 A_2 V_2$. Note the units for the two terms of Eq. 2 (kg/s or slugs/s). Note that if there were both an inlet and an outlet to the control volume shown in Fig. E4.7, Eq. 2 would become

$$\frac{\partial\left(\int_{\text{cv}} \rho \, d\forall\right)}{\partial t} = \rho_1 A_1 V_1 - \rho_2 A_2 V_2 \tag{3}$$

In addition, if the flow were steady, the left-hand side of Eq. 3 would be zero (the amount of mass in the control would be constant in time) and Eq. 3 would become

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

This is one form of the conservation of mass principle—the mass flowrates into and out of the control volume are equal. Other more general forms are discussed in Chapter 5.



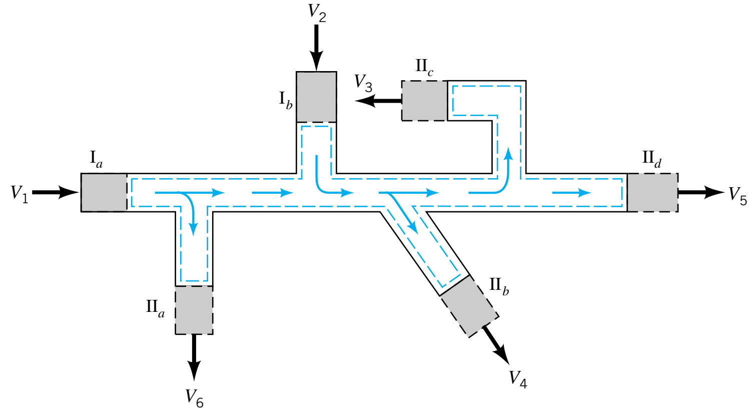
■ **FIGURE 4.12 Control volume and system for flow through an arbitrary, fixed control volume.**

Equation 4.15 is a simplified version of the Reynolds transport theorem. We will now derive it for much more general conditions. A general, fixed control volume with fluid flowing through it is shown in Fig. 4.12. The flow field may be quite simple (as in the above one-dimensional flow considerations), or it may involve a quite complex, unsteady, three-dimensional situation. In any case we again consider the system to be the fluid within the control volume at the initial time t. A short time later a portion of the fluid (region II) has exited from the control volume and additional fluid (region I, not part of the original system) has entered the control volume.

The simplified Reynolds transport theorem can be easily generalized.

We consider an extensive fluid property B and seek to determine how the rate of change of B associated with the system is related to the rate of change of B within the control volume at any instant. By repeating the exact steps that we did for the simplified control volume shown in Fig. 4.11, we see that Eq. 4.14 is valid for the general case also, provided that we give the correct interpretation to the terms $\dot{B}_{\text{out}}$ and $\dot{B}_{\text{in}}$. In general, the control volume may contain more (or less) than one inlet and one outlet. A typical pipe system may contain several inlets and outlets as are shown in Fig. 4.13. In such instances we think of all inlets grouped together ($\text{I} = \text{I}_a + \text{I}_b + \text{I}_c + \cdots$) and all outlets grouped together ($\text{II} = \text{II}_a + \text{II}_b + \text{II}_c + \cdots$), at least conceptually.

The term $\dot{B}_{\text{out}}$ represents the net flowrate of the property B from the control volume. Its value can be thought of as arising from the addition (integration) of the contributions through each infinitesimal area element of size δA on the portion of the control surface



■ **FIGURE 4.13 Typical control volume with more than one inlet and outlet.**

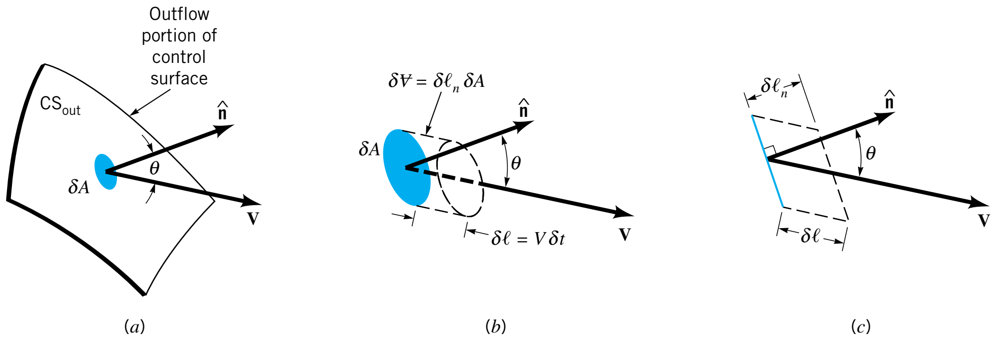


FIGURE 4.14 **Outflow across a typical portion of the control surface.**

dividing region II and the control volume. This surface is denoted $\mathrm{CS_{out}}$. As is indicated in Fig. 4.14, in time δt the volume of fluid that passes across each area element is given by $\delta \forall = \delta\ell_n\, \delta A$, where $\delta\ell_n = \delta\ell \cos\theta$ is the height (normal to the base, δA) of the small volume element, and θ is the angle between the velocity vector and the outward pointing normal to the surface, $\hat{\mathbf{n}}$. Thus, since $\delta\ell = V\,\delta t$, the amount of the property B carried across the area element δA in the time interval δt is given by

$$\delta B = b\rho\,\delta\forall = b\rho(V\cos\theta\,\delta t)\,\delta A$$

The rate at which B is carried out of the control volume across the small area element δA, denoted $\delta\dot{B}_{\mathrm{out}}$, is

$$\delta\dot{B}_{\mathrm{out}} = \lim_{\delta t\to 0}\frac{\rho b\,\delta\forall}{\delta t} = \lim_{\delta t\to 0}\frac{(\rho bV\cos\theta\,\delta t)\,\delta A}{\delta t} = \rho bV\cos\theta\,\delta A$$

By integrating over the entire outflow portion of the control surface, $\mathrm{CS_{out}}$, we obtain

$$\dot{B}_{\mathrm{out}} = \int_{\mathrm{cs_{out}}} d\dot{B}_{\mathrm{out}} = \int_{\mathrm{cs_{out}}} \rho bV\cos\theta\,dA$$

The quantity $V\cos\theta$ is the component of the velocity normal to the area element δA. From the definition of the dot product, this can be written as $V\cos\theta = \mathbf{V}\cdot\hat{\mathbf{n}}$. Hence, an alternate form of the outflow rate is

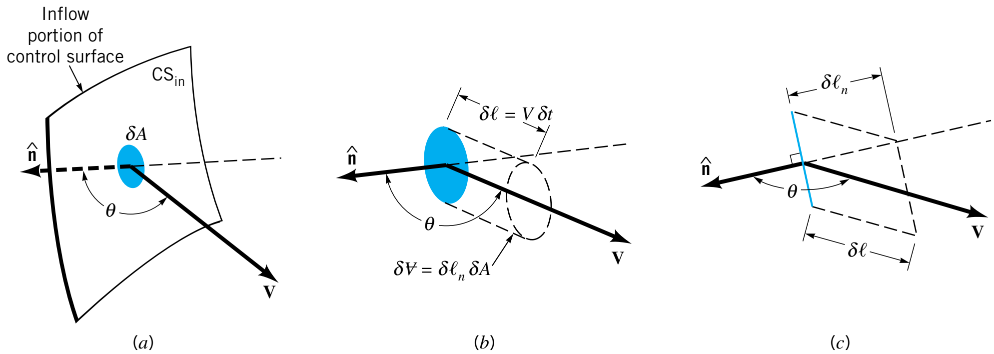
The flowrate of a parameter across the control surface is written in terms of a surface integral.

$$\dot{B}_{\mathrm{out}} = \int_{\mathrm{cs_{out}}} \rho b\mathbf{V}\cdot\hat{\mathbf{n}}\,dA \tag{4.16}$$

In a similar fashion, by considering the inflow portion of the control surface, $\mathrm{CS_{in}}$, as shown in Fig. 4.15, we find that the inflow rate of B into the control volume is

$$\dot{B}_{\mathrm{in}} = -\int_{\mathrm{cs_{in}}} \rho bV\cos\theta\,dA = -\int_{\mathrm{cs_{in}}} \rho b\mathbf{V}\cdot\hat{\mathbf{n}}\,dA \tag{4.17}$$

We use the standard notation that the unit normal vector to the control surface, $\hat{\mathbf{n}}$, points out from the control volume. Thus, as is shown in Fig. 4.16, $-90° < \theta < 90°$ for outflow regions (the normal component of $\mathbf{V}$ is positive; $\mathbf{V}\cdot\hat{\mathbf{n}} > 0$). For inflow regions $90° < \theta < 270°$ (the normal component of $\mathbf{V}$ is negative; $\mathbf{V}\cdot\hat{\mathbf{n}} < 0$). The value of $\cos\theta$ is, therefore, positive on the $\mathrm{CV_{out}}$ portions of the control surface and negative on the $\mathrm{CV_{in}}$ portions. Over the remainder of the control surface, there is no inflow or outflow, leading to $\mathbf{V}\cdot\hat{\mathbf{n}} = V\cos\theta = 0$ on those portions. On such portions either $V = 0$ (the fluid "sticks" to the surface) or $\cos\theta = 0$ (the fluid "slides"along the surface without crossing it) (see Fig.



■ **FIGURE 4.15** **Inflow across a typical portion of the control surface.**

4.16). Therefore, the net flux (flowrate) of parameter B across the entire control surface is

$$\begin{aligned}\dot{B}_{\text{out}} - \dot{B}_{\text{in}} &= \int_{\text{cs}_{\text{out}}} \rho b \mathbf{V} \cdot \hat{\mathbf{n}}\, dA - \left(-\int_{\text{cs}_{\text{in}}} \rho b \mathbf{V} \cdot \hat{\mathbf{n}}\, dA\right) \\ &= \int_{\text{cs}} \rho b \mathbf{V} \cdot \hat{\mathbf{n}}\, dA \end{aligned} \tag{4.18}$$

where the integration is over the entire control surface.

By combining Eqs. 4.14 and 4.18 we obtain

$$\frac{DB_{\text{sys}}}{Dt} = \frac{\partial B_{\text{cv}}}{\partial t} + \int_{\text{cs}} \rho b \mathbf{V} \cdot \hat{\mathbf{n}}\, dA$$

This can be written in a slightly different form by using $B_{\text{cv}} = \int_{\text{cv}} \rho b\, d\mathcal{V}$ so that

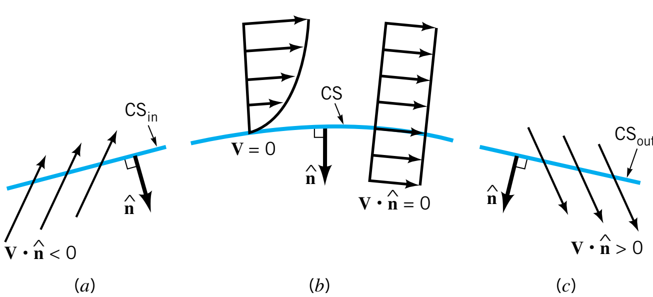
The general Reynolds transport theorem involves volume and surface integrals.

$$\boxed{\frac{DB_{\text{sys}}}{Dt} = \frac{\partial}{\partial t}\int_{\text{cv}} \rho b\, d\mathcal{V} + \int_{\text{cs}} \rho b\, \mathbf{V} \cdot \hat{\mathbf{n}}\, dA} \tag{4.19}$$

Equation 4.19 is the general form of the Reynolds transport theorem for a fixed, nondeforming control volume. Its interpretation and use are discussed in the following sections.

4.4.2 Physical Interpretation

The Reynolds transport theorem as given in Eq. 4.19 is widely used in fluid mechanics (and other areas as well). At first it appears to be a rather formidable mathematical expression—perhaps one to be steered clear of if possible. However, a physical understanding of the concepts involved will show that it is a rather straightforward, relatively easy-to-use tool. Its purpose is to provide a link between control volume ideas and system ideas.



■ **FIGURE 4.16** **Possible velocity configurations on portions of the control surface: (*a*) inflow, (*b*) no flow across the surface, (*c*) outflow.**

The left side of Eq. 4.19 is the time rate of change of an arbitrary extensive parameter of a system. This may represent the rate of change of mass, momentum, energy, or angular momentum of the system, depending on the choice of the parameter B.

Because the system is moving and the control volume is stationary, the time rate of change of the amount of B within the control volume is not necessarily equal to that of the system. The first term on the right side of Eq. 4.19 represents the rate of change of B within the control volume as the fluid flows through it. Recall that b is the amount of B per unit mass, so that $\rho b\, d\forall$ is the amount of B in a small volume $d\forall$. Thus, the time derivative of the integral of ρb throughout the control volume is the time rate of change of B within the control volume at a given time.

The last term in Eq. 4.19 (an integral over the control surface) represents the net flowrate of the parameter B across the entire control surface. Over a portion of the control surface this property is being carried out of the control volume ($\mathbf{V} \cdot \hat{\mathbf{n}} > 0$); over other portions it is being carried into the control volume ($\mathbf{V} \cdot \hat{\mathbf{n}} < 0$). Over the remainder of the control surface there is no transport of B across the surface since $b\mathbf{V} \cdot \hat{\mathbf{n}} = 0$, because either $b = 0$, $\mathbf{V} = 0$, or $\mathbf{V}$ is parallel to the surface at those locations. The mass flowrate through area element δA, given by $\rho \mathbf{V} \cdot \hat{\mathbf{n}}\, \delta A$, is positive for outflow (efflux) and negative for inflow (influx). Each fluid particle or fluid mass carries a certain amount of B with it, as given by the product of B per unit mass, b, and the mass. The rate at which this B is carried across the control surface is given by the area integral term of Eq. 4.19. This net rate across the entire control surface may be negative, zero, or positive depending on the particular situation involved.

4.4.3 Relationship to Material Derivative

In Section 4.2.1 we discussed the concept of the material derivative $D(\ )/Dt = \partial(\ )/\partial t + \mathbf{V} \cdot \nabla(\ ) = \partial(\ )\partial t + u\, \partial(\ )\partial x + v\, \partial(\ )/\partial y + w\, \partial(\ )/\partial z$. The physical interpretation of this derivative is that it provides the time rate of change of a fluid property (temperature, velocity, etc.) associated with a particular fluid particle as it flows. The value of that parameter for that particle may change because of unsteady effects [the $\partial(\ )/\partial t$ term] or because of effects associated with the particle's motion [the $\mathbf{V} \cdot \nabla(\ )$ term].

Careful consideration of Eq. 4.19 indicates the same type of physical interpretation for the Reynolds transport theorem. The term involving the time derivative of the control volume integral represents unsteady effects associated with the fact that values of the parameter within the control volume may change with time. For steady flow this effect vanishes—fluid flows through the control volume but the amount of any property, B, within the control volume is constant in time. The term involving the control surface integral represents the convective effects associated with the flow of the system across the fixed control surface. The sum of these two terms gives the rate of change of the parameter B for the system. This corresponds to the interpretation of the material derivative, $D(\ )/Dt = \partial(\ )/\partial t + \mathbf{V} \cdot \nabla(\ )$, in which the sum of the unsteady effect and the convective effect gives the rate of change of a parameter for a fluid particle. As is discussed in Section 4.2, the material derivative operator may be applied to scalars (such as temperature) or vectors (such as velocity). This is also true for the Reynolds transport theorem. The particular parameters of interest, B and b, may be scalars or vectors.

The Reynolds transport theorem is the integral counterpart of the material derivative.

Thus, both the material derivative and the Reynolds transport theorem equations represent ways to transfer from the Lagrangian viewpoint (follow a particle or follow a system) to the Eulerian viewpoint (observe the fluid at a given location in space or observe what happens in the fixed control volume). The material derivative (Eq. 4.5) is essentially the infinitesimal (or derivative) equivalent of the finite size (or integral) Reynolds transport theorem (Eq. 4.19).

4.4.4 Steady Effects

Consider a steady flow $[\partial(\ )/\partial t \equiv 0]$ so that Eq. 4.19 reduces to

$$\frac{DB_{\text{sys}}}{Dt} = \int_{\text{cs}} \rho b \mathbf{V} \cdot \hat{\mathbf{n}}\, dA \tag{4.20}$$

In such cases if there is to be a change in the amount of B associated with the system (nonzero left-hand side), there must be a net difference in the rate that B flows into the control volume compared with the rate that it flows out of the control volume. That is, the integral of $\rho b \mathbf{V} \cdot \hat{\mathbf{n}}$ over the inflow portions of the control surface would not be equal and opposite to that over the outflow portions of the surface.

Consider steady flow through the "black box" control volume that is shown in Fig. 4.17. If the parameter B is the mass of the system, the left-hand side of Eq. 4.20 is zero (conservation of mass for the system as discussed in detail in **Section 5.1**). Hence, the flowrate of mass into the box must be the same as the flowrate of mass out of the box because the right-hand side of Eq. 4.20 represents the net flowrate through the control surface. On the other hand, assume the parameter B is the momentum of the system. The momentum of the system need not be constant. In fact, according to Newton's second law the time rate of change of the system momentum equals the net force, $\mathbf{F}$, acting on the system. In general, the left-hand side of Eq. 4.20 will therefore be nonzero. Thus, the right-hand side, which then represents the net flux of momentum across the control surface, will be nonzero. The flowrate of momentum into the control volume need not be the same as the flux of momentum from the control volume. We will investigate these concepts much more fully in Chapter 5. They are the basic principles describing the operation of such devices as jet or rocket engines.

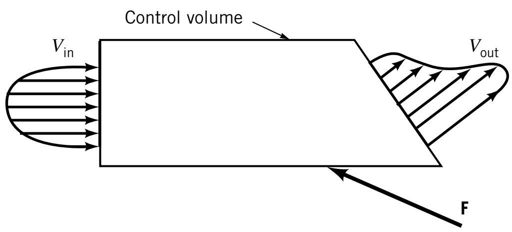
The Reynolds transport theorem involves both steady and unsteady effects.

For steady flows the amount of the property B within the control volume does not change with time. The amount of the property associated with the system may or may not change with time, depending on the particular property considered and the flow situation involved. The difference between that associated with the control volume and that associated with the system is determined by the rate at which B is carried across the control surface—the term $\int_{\text{cs}} \rho b \mathbf{V} \cdot \hat{\mathbf{n}}\, dA$.

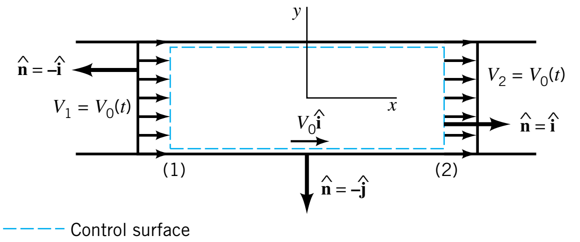
4.4.5 Unsteady Effects

Consider unsteady flow $[\partial(\ )/\partial t \neq 0]$ so that all terms in Eq. 4.19 must be retained. When they are viewed from a control volume standpoint, the amount of parameter B within the system may change because the amount of B within the fixed control volume may change with time $[\text{the } \partial(\int_{\text{cv}} \rho b\, d\mathcal{V})/\partial t \text{ term}]$ and because there may be a net nonzero flow of that parameter across the control surface (the $\int_{\text{cs}} \rho b \mathbf{V} \cdot \hat{\mathbf{n}}\, dA$ term).

For the special unsteady situations in which the rate of inflow of parameter B is exactly balanced by its rate of outflow, it follows that $\int_{\text{cs}} \rho b \mathbf{V} \cdot \hat{\mathbf{n}}\, dA = 0$, and Eq. 4.19 reduces to



■ **FIGURE 4.17 Steady flow through a control volume.**



■ **FIGURE 4.18 Unsteady flow through a constant diameter pipe.**

$$\frac{DB_{\mathrm{sys}}}{Dt} = \frac{\partial}{\partial t}\int_{\mathrm{cv}} \rho b \, d\forall \tag{4.21}$$

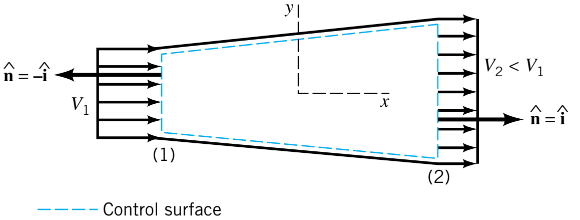
For such cases, any rate of change in the amount of B associated with the system is equal to the rate of change of B within the control volume. This can be illustrated by considering flow through a constant diameter pipe as is shown in Fig. 4.18. The control volume is as shown, and the system is the fluid within this volume at time t_0. We assume the flow is one-dimensional with $\mathbf{V} = V_0\hat{\mathbf{i}}$, where $V_0(t)$ is a function of time, and that the density is constant. At any instant in time, all particles in the system have the same velocity. We let $\mathbf{B}$ =system momentum $= m\mathbf{V} = mV_0\hat{\mathbf{i}}$, where m is the system mass, so that $\mathbf{b} = \mathbf{B}/m = \mathbf{V} = V_0\hat{\mathbf{i}}$, the fluid velocity. The magnitude of the momentum efflux across the outlet [section (2)] is the same as the magnitude of the momentum influx across the inlet [section (1)]. However, the sign of the efflux is opposite to that of the influx since $\mathbf{V}\cdot\hat{\mathbf{n}} > 0$ for the outflow and $\mathbf{V}\cdot\hat{\mathbf{n}} < 0$ for the inflow. Note that $\mathbf{V}\cdot\hat{\mathbf{n}} = 0$ along the sides of the control volume. Thus, with $\mathbf{V}\cdot\hat{\mathbf{n}} = -V_0$ on section (1), $\mathbf{V}\cdot\hat{\mathbf{n}} = V_0$ on section (2), and $A_1 = A_2$, we obtain

In many situations the integrals involved in the Reynolds transport theorem reduce to simple algebra.

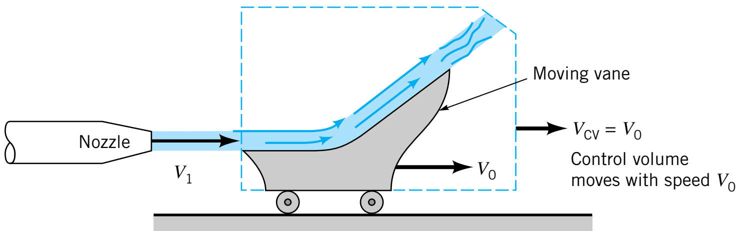
$$\begin{aligned}\int_{\mathrm{cs}} \rho b \mathbf{V}\cdot\hat{\mathbf{n}}\, dA &= \int_{\mathrm{cs}} \rho(V_0\hat{\mathbf{i}})(\mathbf{V}\cdot\hat{\mathbf{n}})\, dA \\ &= \int_{(1)} \rho(V_0\hat{\mathbf{i}})(-V_0)\, dA + \int_{(2)} \rho(V_0\hat{\mathbf{i}})(V_0)\, dA \\ &= -\rho V_0^2 A_1\hat{\mathbf{i}} + \rho V_0^2 A_2\hat{\mathbf{i}} = 0\end{aligned}$$

It is seen that for this special case Eq. 4.21 is valid. The rate at which the momentum of the system changes with time is the same as the rate of change of momentum within the control volume. If V_0 is constant in time, there is no rate of change of momentum of the system and for this special case each of the terms in the Reynolds transport theorem is zero by itself.

Consider the flow through a variable area pipe shown in Fig. 4.19. In such cases the fluid velocity is not the same at section (1) as it is at (2). Hence, the efflux of momentum from the control volume is not equal to the influx of momentum, so that the convective term in Eq. 4.20 [the integral of $\rho\mathbf{V}(\mathbf{V}\cdot\hat{\mathbf{n}})$ over the control surface] is not zero. These topics will be discussed in considerably more detail in Chapter 5.



■ **FIGURE 4.19 Flow through a variable area pipe.**



■ FIGURE 4.20 **Example of a moving control volume.**

4.4.6 Moving Control Volumes

Some problems are most easily solved by using a moving control volume.

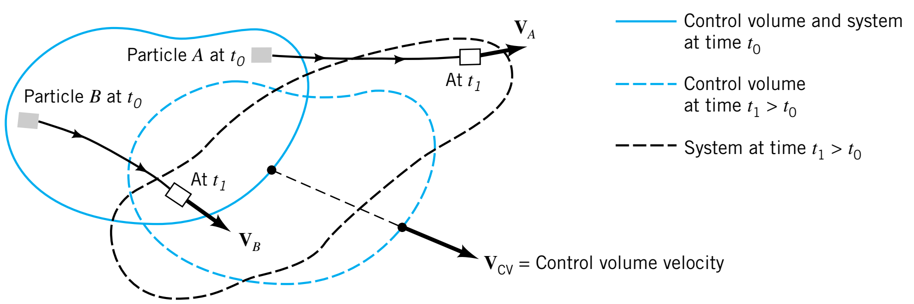
For most problems in fluid mechanics, the control volume may be considered as a fixed volume through which the fluid flows. There are, however, situations for which the analysis is simplified if the control volume is allowed to move or deform. The most general situation would involve a control volume that moves, accelerates, and deforms. As one might expect, the use of these control volumes can become fairly complex.

A number of important problems can be most easily analyzed by using a nondeforming control volume that moves with a constant velocity. Such an example is shown in Fig. 4.20 in which a stream of water with velocity $\mathbf{V}_1$ strikes a vane that is moving with constant velocity $\mathbf{V}_0$. It may be of interest to determine the force, $\mathbf{F}$, that the water puts on the vane. Such problems frequently occur in turbines where a stream of fluid (water or steam, for example) strikes a series of blades that move past the nozzle. To analyze such problems it is advantageous to use a moving control volume. We will obtain the Reynolds transport theorem for such control volumes.

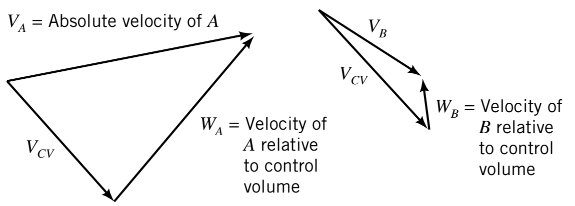
We consider a control volume that moves with a constant velocity as is shown in Fig. 4.21. The shape, size, and orientation of the control volume do not change with time. The control volume merely translates with a constant velocity, $\mathbf{V}_{cv}$, as shown. In general, the velocity of the control volume and the fluid are not the same, so that there is a flow of fluid through the moving control volume just as in the stationary control volume cases discussed in Section 4.4.2. The main difference between the fixed and the moving control volume cases is that it is the *relative velocity,* $\mathbf{W}$, that carries fluid across the moving control surface, whereas it is the *absolute velocity,* $\mathbf{V}$, that carries the fluid across the fixed control surface. The relative velocity is the fluid velocity relative to the moving control volume—the fluid velocity seen by an observer riding along on the control volume. The absolute velocity is the fluid velocity as seen by a stationary observer in a fixed coordinate system.

The difference between the absolute and relative velocities is the velocity of the control volume, $\mathbf{V}_{cv} = \mathbf{V} - \mathbf{W}$, or

$$\mathbf{V} = \mathbf{W} + \mathbf{V}_{cv} \tag{4.22}$$



■ FIGURE 4.21 **Typical moving control volume and system.**



■ **FIGURE 4.22**
Relationship between absolute and relative velocities.

Since the velocity is a vector, we must use vector addition as is shown in Fig. 4.22 to obtain the relative velocity if we know the absolute velocity and the velocity of the control volume. Thus, if the water leaves the nozzle in Fig. 4.20 with a velocity of $\mathbf{V}_1 = 100\hat{\mathbf{i}}$ ft/s and the vane has a velocity of $\mathbf{V}_0 = 20\hat{\mathbf{i}}$ ft/s (the same as the control volume), it appears to an observer riding on the vane that the water approaches the vane with a velocity of $\mathbf{W} = \mathbf{V} - \mathbf{V}_{\text{cv}} = 80\hat{\mathbf{i}}$ ft/s. In general, the absolute velocity, $\mathbf{V}$, and the control volume velocity, $\mathbf{V}_{\text{cv}}$, will not be in the same direction so that the relative and absolute velocities will have different directions (see Fig. 4.22).

The Reynolds transport theorem for a moving, nondeforming control volume can be derived in the same manner that it was obtained for a fixed control volume. As is indicated in Fig. 4.23, the only difference that needs be considered is the fact that relative to the moving control volume the fluid velocity observed is the relative velocity, not the absolute velocity. An observer fixed to the moving control volume may or may not even know that he or she is moving relative to some fixed coordinate system. If we follow the derivation that led to Eq. 4.19 (the Reynolds transport theorem for a fixed control volume), we note that the corresponding result for a moving control volume can be obtained by simply replacing the absolute velocity, $\mathbf{V}$, in that equation by the relative velocity, $\mathbf{W}$. Thus, the Reynolds transport theorem for a control volume moving with constant velocity is given by

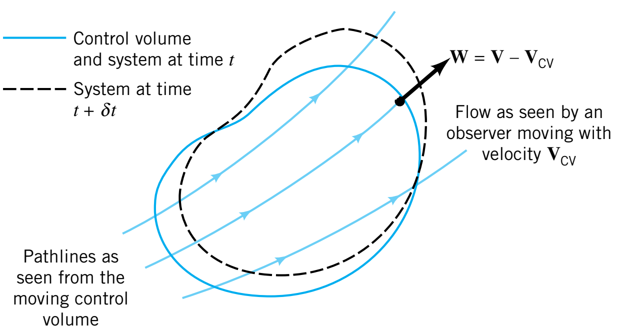
The Reynolds transport theorem for a moving control volume involves the relative velocity.

$$\frac{DB_{\text{sys}}}{Dt} = \frac{\partial}{\partial t}\int_{\text{cv}} \rho b \, d\forall + \int_{\text{cs}} \rho b \, \mathbf{W} \cdot \hat{\mathbf{n}} \, dA \tag{4.23}$$

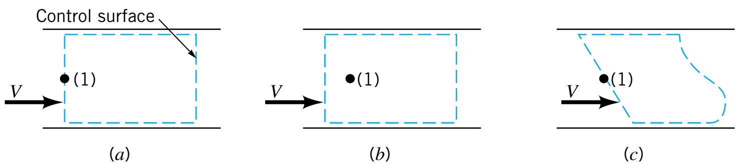
where the relative velocity is given by Eq. 4.22.

4.4.7 Selection of a Control Volume

Any volume in space can be considered as a control volume. It may be of finite size or it may be infinitesimal in size, depending on the type of analysis to be carried out. In most of our cases, the control volume will be a fixed, nondeforming volume. In some situations we will consider control volumes that move with constant velocity. In either case it is important that considerable thought go into the selection of the specific control volume to be used.



■ **FIGURE 4.23**
Control volume and system as seen by an observer moving with the control volume.



■ **FIGURE 4.24 Various control volumes for flow through a pipe.**

For any problem there is an infinite variety of control volumes that can be used.

The selection of an appropriate control volume in fluid mechanics is very similar to the selection of an appropriate free-body diagram in dynamics or statics. In dynamics, we select the body in which we are interested, represent the object in a free-body diagram, and then apply the appropriate governing laws to that body. The ease of solving a given dynamics problem is often very dependent on the specific object that we select for use in our free-body diagram. Similarly, the ease of solving a given fluid mechanics problem is often very dependent on the choice of the control volume used. Only by practice can we develop skill at selecting the "best" control volume. None are "wrong," but some are "much better" than others.

Solution of a typical problem will involve determining parameters such as velocity, pressure, and force at some point in the flow field. It is usually best to ensure that this point is located on the control surface, not "buried" within the control volume. The unknown will then appear in the convective term (the surface integral) of the Reynolds transport theorem. If possible, the control surface should be normal to the fluid velocity so that the angle θ ($\mathbf{V} \cdot \hat{\mathbf{n}} = V \cos \theta$) in the flux terms of Eq. 4.19 will be 0 or 180°. This will usually simplify the solution process.

Figure 4.24 illustrates three possible control volumes associated with flow through a pipe. If the problem is to determine the pressure at point (1), the selection of the control volume (*a*) is better than that of (*b*) because point (1) lies on the control surface. Similarly, control volume (*a*) is better than (*c*) because the flow is normal to the inlet and exit portions of the control volume. None of these control volumes are wrong—(*a*) will be easier to use. Proper control volume selection will become much clearer in Chapter 5 where the Reynolds transport theorem is used to transform the governing equations from the system formulation into the control volume formulation, and numerous examples using control volume ideas are discussed.

Key Words and Topics

In the E-book, click on any key word or topic to go to that subject.

Acceleration field
Control volume
Convective acceleration
Convective derivative
Eulerian description
Extensive property
Field representation
Intensive property
Lagrangian description
Local acceleration
Material derivative
Moving control volume
One-, two-, three-dimensional flow
Pathlines
Reynolds transport theorem
Steady flow
Streaklines
Streamlines
System
Unsteady flow
Velocity field

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2. Goldstein, R. J., *Fluid Mechanics Measurements*, Hemisphere, New York, 1983.

3. Homsy, G. M., et al., *Multimedia Fluid Mechanics* CD-ROM, Cambridge University Press, New York, 2000.

4. Magarvey, R. H., and MacLatchy, C. S., The Formation and Structure of Vortex Rings, *Canadian Journal of Physics*, Vol. 42, 1964.

Review Problems

In the E-book, *click here* to go to a set of review problems complete with answers and detailed solutions.

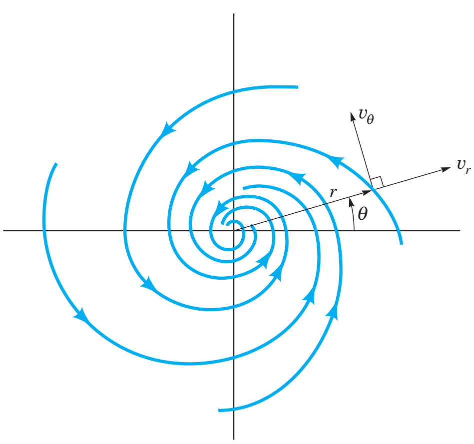
Problems

Note: Unless otherwise indicated, use the values of fluid properties found in the tables on the inside of the front cover. Problems designated with an (*) are intended to be solved with the aid of a programmable calculator or a computer. Problems designated with a (†) are "open-ended" problems and require critical thinking in that to work them one must make various assumptions and provide the necessary data. There is not a unique answer to these problems.

In the E-book, answers to the even-numbered problems can be obtained by clicking on the problem number. In the E-book, access to the videos that accompany problems can be obtained by clicking on the "video" segment (i.e., **Video 4.3**) of the problem statement. The lab-type problems can be accessed by clicking on the "*click here*" segment of the problem statement.

4.1 The velocity field of a flow is given by $\mathbf{V} = (3y + 2)\hat{\mathbf{i}} + (x - 8)\hat{\mathbf{j}} + 5z\hat{\mathbf{k}}$ ft/s, where x, y, and z are in feet. Determine the fluid speed at the origin ($x = y = z = 0$) and on the y axis ($x = z = 0$).

4.2 A flow can be visualized by plotting the velocity field as velocity vectors at representative locations in the flow as shown in **Video V4.1** and Fig. E4.1. Consider the velocity field given in polar coordinates by $v_r = -10/r$ and $v_\theta = 10/r$. This flow approximates a fluid swirling into a sink as shown in Fig. P4.2. Plot the velocity field at locations given by $r = 1$, 2, and 3 with $\theta = 0$, 30, 60, and 90 deg.



■ **FIGURE P4.2**

4.3 The velocity field of a flow is given by $\mathbf{V} = 20y/(x^2 + y^2)^{1/2}\hat{\mathbf{i}} - 20x/(x^2 + y^2)^{1/2}\hat{\mathbf{j}}$ ft/s, where x and y are in feet. Determine the fluid speed at points along the x axis; along the y axis. What is the angle between the velocity vector and the x axis at points $(x, y) = (5, 0)$, $(5, 5)$, and $(0, 5)$?

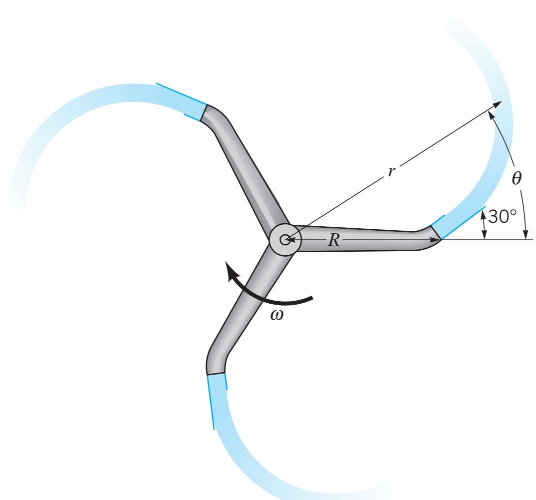
4.4 The x and y components of a velocity field are given by $u = x - y$ and $v = x^2y - 8$. Determine the location of any stagnation points in the flow field. That is, at what point(s) is the velocity zero?

4.5 The x and y components of velocity for a two-dimensional flow are $u = 3$ ft/s and $v = 9x^2$ ft/s, where x is in feet. Determine the equation for the streamlines and graph representative streamlines in the upper half plane.

4.6 Show that the streamlines for a flow whose velocity components are $u = c(x^2 - y^2)$ and $v = -2cxy$, where c is a constant, are given by the equation $x^2y - y^3/3 =$ constant. At which point (points) is the flow parallel to the y axis? At which point (points) is the fluid stationary?

4.7 The velocity field of a flow is given by $u = -V_0y/(x^2 + y^2)^{1/2}$ and $v = V_0x/(x^2 + y^2)^{1/2}$, where V_0 is a constant. Where in the flow field is the speed equal to V_0? Determine the equation of the streamlines and discuss the various characteristics of this flow.

4.8 Water flows from a rotating lawn sprinkler as shown in **Video V4.6** and Figure P4.8. The end of the sprinkler arm moves with a speed of ωR, where $\omega = 10$ rad/s is the angular velocity of the sprinkler arm and $R = 0.5$ ft is its radius. The water exits the nozzle with a speed of $V = 10$ ft/s relative to the rotating arm. Gravity and the interaction between the air and the water are negligible. **(a)** Show that the pathlines for this flow are straight radial lines. *Hint:* Consider the direction of flow (relative to the stationary ground) as the water leaves the sprinkler arm. **(b)** Show that at any given instant the stream of water that came from the sprinkler forms an arc given by $r = R + (V_a/\omega)\theta$, where the angle θ is as indicated in the figure and V_a is the water speed relative to the ground. Plot this curve for the data given.



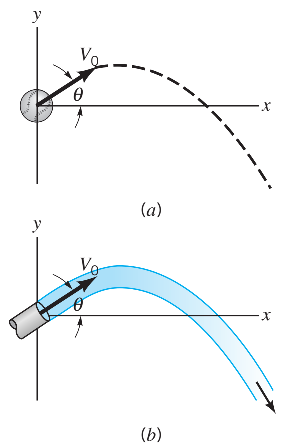
■ **FIGURE P4.8**

***4.9** Consider a ball thrown with initial speed V_0 at an angle of θ as shown in Fig. P4.9*a*. As discussed in beginning physics, if friction is negligible the path that the ball takes is given by

$$y = (\tan\theta)x - [g/(2\,V_0^2\cos^2\theta)]x^2$$

That is, $y = c_1x + c_2x^2$, where c_1 and c_2 are constants. The path is a parabola. The pathline for a stream of water leaving a small nozzle is shown in Fig. P4.9*b* and **Video V4.3**. The coordinates for this water stream are given in the following table. **(a)** Use the given data to determine appropriate values for c_1 and c_2 in the above equation and, thus, show that these water particles also follow a parabolic pathline. **(b)** Use your values of c_1 and c_2 to determine the speed of the water, V_0, leaving the nozzle.

***x*, in.**	***y*, in.**
0	0
0.25	0.13
0.50	0.16
0.75	0.13
1.0	0.00
1.25	−0.20
1.50	−0.53
1.75	−0.90
2.00	−1.43



■ **FIGURE P4.9**

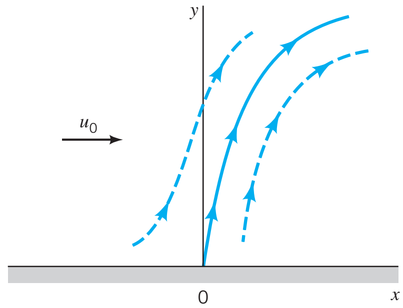
4.10 The x and y components of a velocity field are given by $u = x^2y$ and $v = -xy^2$. Determine the equation for the streamlines of this flow and compare it with those in Example 4.2. Is the flow in this problem the same as that in Example 4.2? Explain.

† **4.11** For steady flow the velocity field, as represented by the velocity vector arrows, is constant (see Fig. E4.1). For unsteady flow the velocity vectors usually change both in magnitude and direction (see **Video V4.1**). However, this is not always so. Describe an unsteady flow field in which the velocity vectors do not change direction as a function of time. Sketch the velocity field for this flow.

4.12 In addition to the customary horizontal velocity components of the air in the atmosphere (the "wind"), there often are vertical air currents (thermals) caused by buoyant effects due to uneven heating of the air as indicated in Fig. P4.12. Assume that the velocity field in a certain region is approximated by $u = u_0$, $v = v_0(1 - y/h)$ for $0 < y < h$, and $u = u_0$, $v = 0$ for $y > h$. Plot the shape of the streamline that passes through the origin for values of $u_0/v_0 = 0.5$, 1, and 2.

***4.13** Repeat Problem 4.12 using the same information except that $u = u_0y/h$ for $0 \le y \le h$ rather than $u = u_0$. Use values of $u_0/v_0 = 0$, 0.1, 0.2, 0.4, 0.6, 0.8, and 1.0.

4.14 A velocity field is given by $u = cx^2$ and $v = cy^2$, where c is a constant. Determine the x and y components of the acceleration. At what point (points) in the flow field is the acceleration zero?

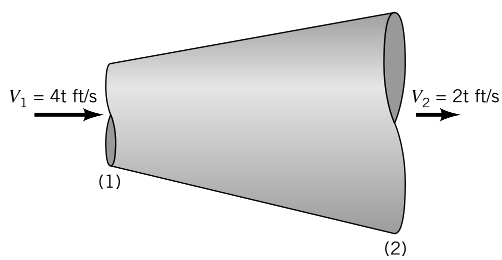


■ **FIGURE P4.12**

4.15 A three-dimensional velocity field is given by $u = x^2$, $v = -2xy$, and $w = x + y$. Determine the acceleration vector.

† **4.16** Estimate the deceleration of a water particle in a raindrop as it strikes the sidewalk. List all assumptions and show all calculations.

4.17 The velocity of air in the diverging pipe shown in Fig. P4.17 is given by $V_1 = 4t$ ft/s and $V_2 = 2t$ ft/s, where t is in seconds. **(a)** Determine the local acceleration at points (1) and (2). **(b)** Is the average convective acceleration between these two points negative, zero, or positive? Explain.



■ **FIGURE P4.17**

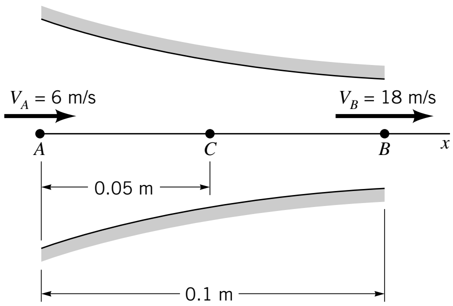
4.18 Water flows through a constant diameter pipe with a uniform velocity given by $\mathbf{V} = (8/t + 5)\hat{\mathbf{j}}$ m/s, where t is in seconds. Determine the acceleration at time $t = 1$, 2, and 10 s.

4.19 When a valve is opened, the velocity of water in a certain pipe is given by $u = 10(1 - e^{-t})$, $v = 0$, and $w = 0$, where u is in ft/s and t is in seconds. Determine the maximum velocity and maximum acceleration of the water.

***4.20** Water flows through a pipe with $\mathbf{V} = u(t)\hat{\mathbf{i}}$ where the approximate measured values of $u(t)$ are shown in the table. Plot the acceleration as a function of time for $0 \le t \le 20$ s. Plot the acceleration as a function of time if all of the values of $u(t)$ are increased by a factor of 2; by a factor of 5.

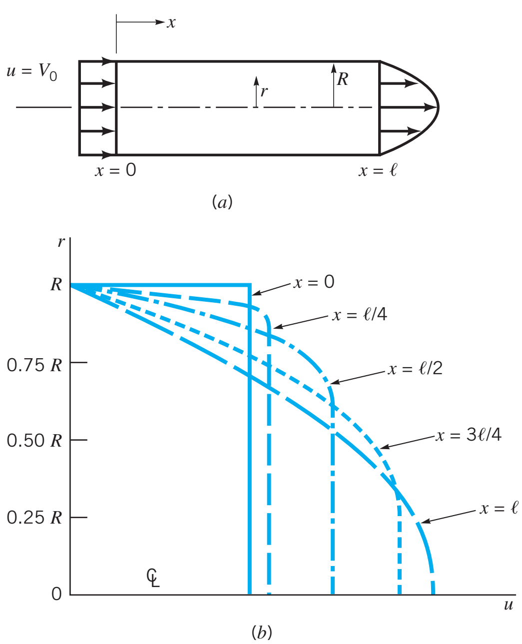
***t* (s)**	***u* (ft/s)**	***t* (s)**	***u* (ft/s)**
0	0	11.2	8.1
1.8	1.7	12.3	8.4
3.1	3.2	13.9	8.3
4.0	3.8	15.0	8.1
5.5	4.6	16.4	7.9
6.9	5.8	17.5	7.0
8.1	6.3	18.4	6.6
10.0	7.1	20.0	5.7

4.21 The fluid velocity along the x axis shown in Fig. P4.21 changes from 6 m/s at point A to 18 m/s at point B. It is also known that the velocity is a linear function of distance along the streamline. Determine the acceleration at points A, B, and C. Assume steady flow.



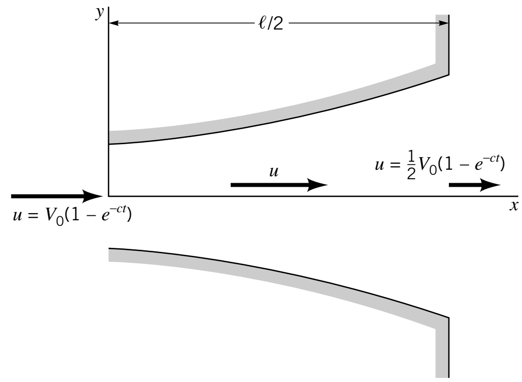
■ **FIGURE P4.21**

4.22 When a fluid flows into a round pipe as shown in Fig. P4.22, viscous effects may cause the velocity profile to change from a uniform profile ($\mathbf{V} = V_0\,\hat{\mathbf{i}}$) at the entrance of the pipe to a parabolic profile $\{\mathbf{V} = 2V_0[1 - (r/R)^2]\,\hat{\mathbf{i}}\}$ at $x = \ell$. Velocity profiles for various values of x are as indicated in the figure. Use this graph to show that a fluid particle moving along the centerline ($r = 0$) experiences an acceleration, but a particle close to the edge of the pipe ($r \approx R$) experiences a deceleration. Does a particle traveling along the line $r = 0.5$ R experience an acceleration or deceleration, or both? Explain.



■ **FIGURE P4.22**

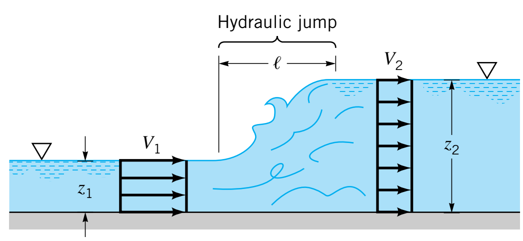
4.23 As a valve is opened, water flows through the diffuser shown in Fig. P4.23 at an increasing flowrate so that the velocity along the centerline is given by $\mathbf{V} = u\hat{\mathbf{i}} = V_0(1 - e^{-ct})(1 - x/\ell)\,\hat{\mathbf{i}}$, where u_0, c, and ℓ are constants. Determine the acceleration as a function of x and t. If $V_0 = 10$ ft/s and $\ell = 5$ ft, what value of c (other than $c = 0$) is needed to make the acceleration zero for any x at $t = 1$ s? Explain how the acceleration can be zero if the flowrate is increasing with time.



■ **FIGURE P4.23**

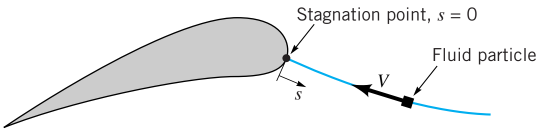
4.24 A fluid flows along the x axis with a velocity given by $\mathbf{V} = (x/t)\,\hat{\mathbf{i}}$, where x is in feet and t in seconds. **(a)** Plot the speed for $0 \le x \le 10$ ft and $t = 3$ s. **(b)** Plot the speed for $x = 7$ ft and $2 \le t \le 4$ s. **(c)** Determine the local and convective acceleration. **(d)** Show that the acceleration of any fluid particle in the flow is zero. **(e)** Explain physically how the velocity of a particle in this unsteady flow remains constant throughout its motion.

4.25 A hydraulic jump is a rather sudden change in depth of a liquid layer as it flows in an open channel as shown in Fig. P4.25 and **Video V10.6**. In a relatively short distance (thickness $= \ell$) the liquid depth changes from z_1 to z_2, with a corresponding change in velocity from V_1 to V_2. If $V_1 = 1.20$ ft/s, $V_2 = 0.30$ ft/s, and $\ell = 0.02$ ft, estimate the average deceleration of the liquid as it flows across the hydraulic jump. How many g's deceleration does this represent?



■ **FIGURE P4.25**

4.26 A fluid particle flowing along a stagnation streamline, as shown in **Video V4.5** and Fig. P4.26, slows down as it approaches the stagnation point. Measurements of the dye flow in the video indicate that the location of a particle starting on the stagnation streamline a distance $s = 0.6$ ft upstream of the stagnation point at $t = 0$ is given approximately by $s = 0.6e^{-0.5t}$, where t is in seconds and s is in ft. **(a)** Determine the speed of a fluid particle as a function of time, $V_{\text{particle}}(t)$, as it flows along the steamline. **(b)** Determine the speed of the fluid as a function of position along the streamline, $V = V(s)$. **(c)** Determine the fluid acceleration along the streamline as a function of position, $a_s = a_s(s)$.



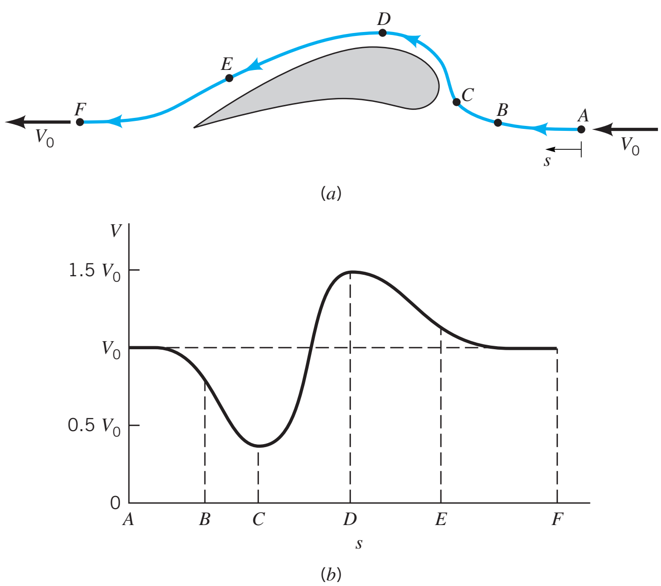
■ FIGURE P4.26

4.27 A nozzle is designed to accelerate the fluid from V_1 to V_2 in a linear fashion. That is, $V = ax + b$, where a and b are constants. If the flow is constant with $V_1 = 10$ m/s at $x_1 = 0$ and $V_2 = 25$ m/s at $x_2 = 1$ m, determine the local acceleration, the convective acceleration, and the acceleration of the fluid at points (1) and (2).

† **4.28** A stream of water from the faucet strikes the bottom of the sink. Estimate the maximum acceleration experienced by the water particles. List all assumptions and show calculations.

4.29 Repeat Problem 4.27 with the assumption that the flow is not steady, but at the time when $V_1 = 10$ m/s and $V_2 = 25$ m/s, it is known that $\partial V_1/\partial t = 20$ m/s^2 and $\partial V_2/\partial t = 60$ m/s^2.

4.30 An incompressible fluid flows past a turbine blade as shown in Fig. P4.30a and **Video V4.5**. Far upstream and downstream of the blade the velocity is V_0. Measurements show that the velocity of the fluid along streamline A–F near the blade is as indicated in Fig. P4.30b. Sketch the streamwise component of acceleration, a_s, as a function of distance, s, along the streamline. Discuss the important characteristics of your result.



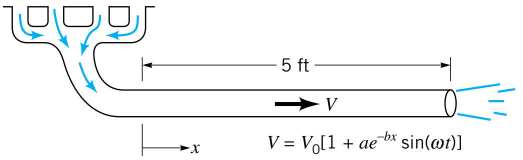
■ FIGURE P4.30

***4.31** Air flows steadily through a variable area pipe with a velocity of $\mathbf{V} = u(x)\hat{\mathbf{i}}$ ft/s, where the approximate measured values of $u(x)$ are given in the table. Plot the acceleration as a function of x for $0 \le x \le 12$ in. Plot the acceleration if the flowrate is increased by a factor of N (i.e., the values of u are increased by a factor of N) for $N = 2, 4, 10$.

x (in.)	u (ft/s)	x (in.)	u (ft/s)
0	10.0	7	20.1
1	10.2	8	17.4
2	13.0	9	13.5
3	20.1	10	11.9
4	28.3	11	10.3
5	28.4	12	10.0
6	25.8	13	10.0

4.32 Assume the temperature of the exhaust in an exhaust pipe can be approximated by $T = T_0(1 + ae^{-bx})[1 + c\cos(\omega t)]$, where $T_0 = 100$ °C, $a = 3$, $b = 0.03$ m^{-1}, $c = 0.05$, and $\omega = 100$ rad/s. If the exhaust speed is a constant 2 m/s, determine the time rate of change of temperature of the fluid particles at $x = 0$ and $x = 4$ m when $t = 0$.

***4.33** As is indicated in Fig. P4.33, the speed of exhaust in a car's exhaust pipe varies in time and distance because of the periodic nature of the engine's operation and the damping effect with distance from the engine. Assume that the speed is given by $V = V_0[1 + ae^{-bx}\sin(\omega t)]$, where $V_0 = 8$ fps, $a = 0.05$, $b = 0.2$ ft^{-1}, and $\omega = 50$ rad/s. Calculate and plot the fluid acceleration at $x = 0, 1, 2, 3, 4$, and 5 ft for $0 \le t \le \pi/25$ s.

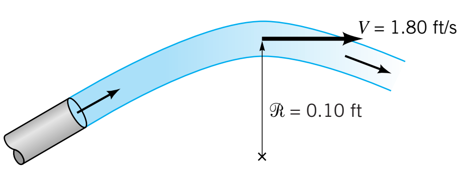


■ FIGURE P4.33

4.34 A gas flows along the x-axis with a speed of $V = 5x$ m/s and a pressure of $p = 10x^2$ N/m^2, where x is in meters. **(a)** Determine the time rate of change of pressure at the fixed location $x = 1$. **(b)** Determine the time rate of change of pressure for a fluid particle flowing past $x = 1$. **(c)** Explain without using any equations why the answers to parts (a) and (b) are different.

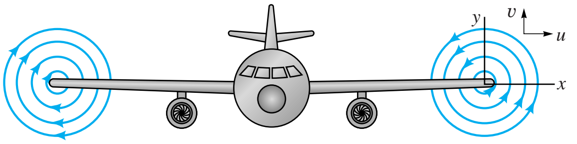
4.35 The temperature distribution in a fluid is given by $T = 10x + 5y$, where x and y are the horizontal and vertical coordinates in meters and T is in degrees centigrade. Determine the time rate of change of temperature of a fluid particle traveling (a) horizontally with $u = 20$ m/s, $v = 0$ or (b) vertically with $u = 0$, $v = 20$ m/s.

4.36 At the top of its trajectory, the stream of water shown in Fig. P4.36 and **Video V4.3** flows with a horizontal velocity of 1.80 ft/s. The radius of curvature of its streamline at that point is approximately 0.10 ft. Determine the normal component of acceleration at that location.



■ FIGURE P4.36

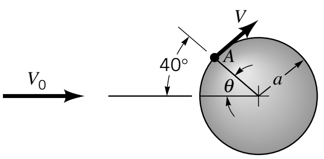
4.37 As shown in **Video V4.2** and Fig. P4.37, a flying airplane produces swirling flow near the end of its wings. In certain circumstances this flow can be approximated by the velocity field $u = -Ky/(x^2 + y^2)$ and $v = Kx/(x^2 + y^2)$, where K is a constant depending on various parameter associated with the airplane (i.e., its weight, speed) and x and y are measured from the center of the swirl. **(a)** Show that for this flow the velocity is inversely proportional to the distance from the origin. That is, $V = K/(x^2 + y^2)^{1/2}$. **(b)** Show that the streamlines are circles.



■ FIGURE P4.37

4.38 Assume that the streamlines for the wingtip vortices from an airplane (see Fig. P4.37 and **Video V4.2**) can be approximated by circles of radius r and that the speed is $V = K/r$, where K is a constant. Determine the streamline acceleration, a_s, and the normal acceleration, a_n, for this flow.

4.39 A fluid flows past a sphere with an upstream velocity of $V_0 = 40$ m/s as shown in Fig. P4.39. From a more advanced theory it is found that the speed of the fluid along the front part of the sphere is $V = \frac{3}{2}V_0 \sin\theta$. Determine the streamwise and normal components of acceleration at point A if the radius of the sphere is $a = 0.20$ m.

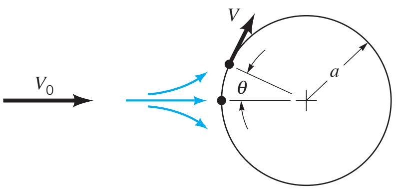


■ FIGURE P4.39

***4.40** For flow past a sphere as discussed in Problem 4.39, plot a graph of the streamwise acceleration, a_s, the normal acceleration, a_n, and the magnitude of the acceleration as a function of θ for $0 \leq \theta \leq 90°$ with $V_0 = 50$ ft/s and $a = 0.1$, 1.0, and 10 ft. Repeat for $V_0 = 5$ ft/s. At what point is the acceleration a maximum; a minimum?

4.41 A fluid flows past a circular cylinder of radius a with an upstream speed of V_0 as shown in Fig. P4.41. A more advanced theory indicates that if viscous effects are negligible, the velocity of the fluid along the surface of the cylinder is given by $V = 2V_0 \sin\theta$. Determine the streamline and normal components of acceleration on the surface of the cylinder as a function of V_0, a, and θ.

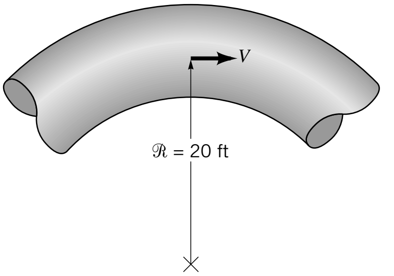
***4.42** Use the results of Problem 4.41 to plot graphs of a_s and a_n for $0 \leq \theta \leq 90°$ with $V_0 = 10$ m/s and $a = 0.01$, 0.10, 1.0, and 10.0 m.



■ FIGURE P4.41

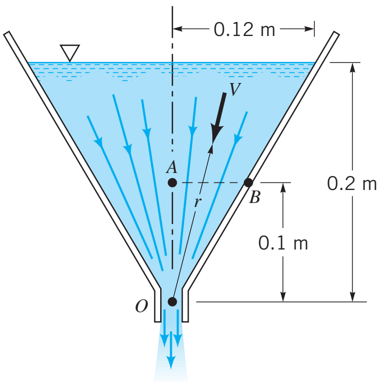
4.43 Determine the x and y components of acceleration for the flow given in Problem 4.6. If $c > 0$, is the particle at point $x = x_0 > 0$ and $y = 0$ accelerating or decelerating? Explain. Repeat if $x_0 < 0$.

4.44 Water flows through the curved hose shown in Fig. P4.44 with an increasing speed of $V = 10t$ ft/s, where t is in seconds. For $t = 2$ s determine **(a)** the component of acceleration along the streamline, **(b)** the component of acceleration normal to the streamline, and **(c)** the net acceleration (magnitude and direction).



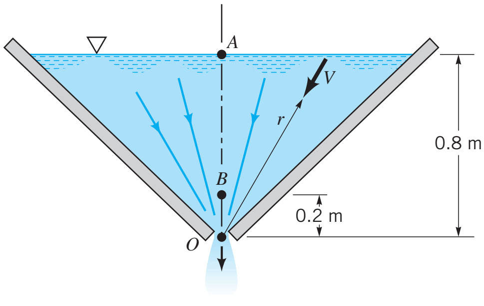
■ FIGURE P4.44

4.45 Water flows steadily through the funnel shown in Fig. P4.45. Throughout most of the funnel the flow is approximately radial (along rays from O) with a velocity of $V = c/r^2$, where r is the radial coordinate and c is a constant. If the velocity is 0.4 m/s when $r = 0.1$ m, determine the acceleration at points A and B.



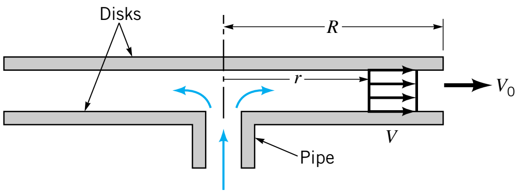
■ FIGURE P4.45

4.46 Water flows though the slit at the bottom of a two-dimensional water trough as shown in Fig. P4.46. Throughout most of the trough the flow is approximately radial (along rays from O) with a velocity of $V = c/r$, where r is the radial coordinate and c is a constant. If the velocity is 0.04 m/s when $r = 0.1$ m, determine the acceleration at points A and B.



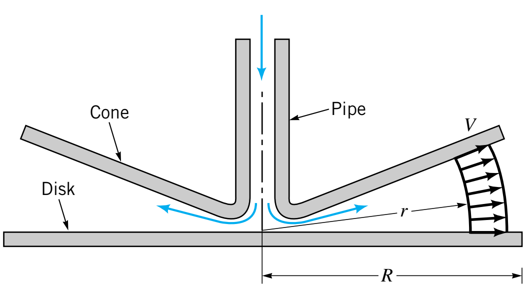
■ **FIGURE P4.46**

4.47 Air flows from a pipe into the region between two parallel circular disks as shown in Fig. P4.47. The fluid velocity in the gap between the disks is closely approximated by $V = V_0 R/r$, where R is the radius of the disk, r is the radial coordinate, and V_0 is the fluid velocity at the edge of the disk. Determine the acceleration for $r = 1$, 2, or 3 ft if $V_0 = 5$ ft/s and $R = 3$ ft.



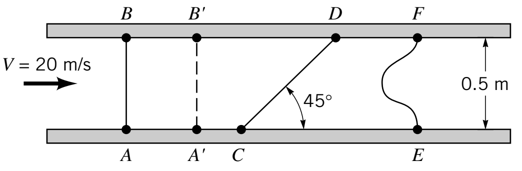
■ **FIGURE P4.47**

4.48 Air flows from a pipe into the region between a circular disk and a cone as shown in Fig. P4.48. The fluid velocity in the gap between the disk and the cone is closely approximated by $V = V_0 R^2/r^2$, where R is the radius of the disk, r is the radial coordinate, and V_0 is the fluid velocity at the edge of the disk. Determine the acceleration for $r = 0.5$ and 2 ft if $V_0 = 5$ ft/s and $R = 2$ ft.



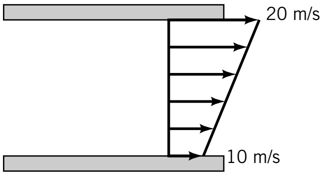
■ **FIGURE P4.48**

4.49 Water flows through a duct of square cross section as shown in Fig. P4.49 with a constant, uniform velocity of $V = 20$ m/s. Consider fluid particles that lie along line A–B at time $t = 0$. Determine the position of these particles, denoted by line $A'-B'$, when $t = 0.20$ s. Use the volume of fluid in the region between lines A–B and $A'-B'$ to determine the flowrate in the duct. Repeat the problem for fluid particles originally along line C–D; along line E–F. Compare your three answers.



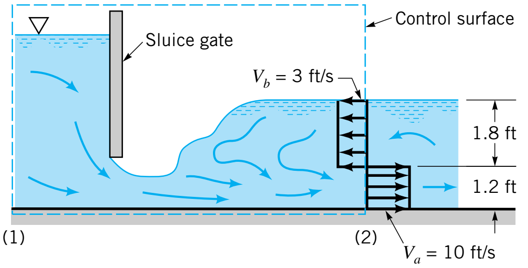
■ **FIGURE P4.49**

4.50 Repeat Problem 4.49 if the velocity profile is linear from 10 to 20 m/s across the duct as shown in Fig. P4.50.



■ **FIGURE P4.50**

4.51 In the region just downstream of a sluice gate, the water may develop a reverse flow region as is indicated in Fig. P4.51 and **Video V10.5**. The velocity profile is assumed to consist of two uniform regions, one with velocity $V_a = 10$ fps and the other with $V_b = 3$ fps. Determine the net flowrate of water across the portion of the control surface at section (2) if the channel is 20 ft wide.



■ **FIGURE P4.51**

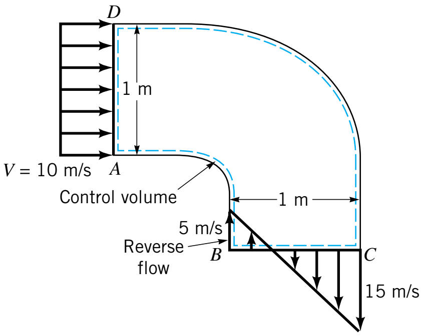
4.52 At time $t = 0$ the valve on an initially empty (perfect vacuum, $\rho = 0$) tank is opened and air rushes in. If the tank has a volume of $\mathcal{V}_0$ and the density of air within the tank increases as $\rho = \rho_\infty(1 - e^{-bt})$, where b is a constant, determine the time rate of change of mass within the tank.

† **4.53** From calculus, one obtains the following formula (Leibnitz rule) for the time derivative of an integral that contains time in both the integrand and the limits of the integration:

$$\frac{d}{dt}\int_{x_1(t)}^{x_2(t)} f(x,t)dx = \int_{x_1}^{x_2} \frac{\partial f}{\partial t}dx + f(x_2,t)\frac{dx_2}{dt} - f(x_1,t)\frac{dx_1}{dt}$$

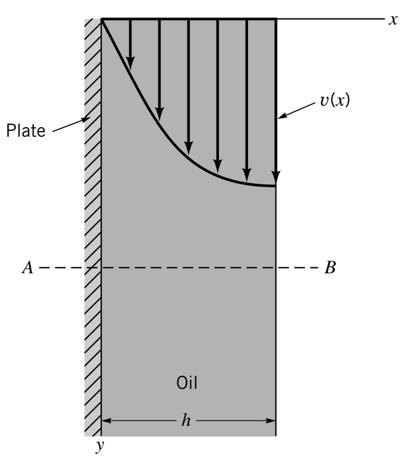
Discuss how this formula is related to the time derivative of the total amount of a property in a system and to the Reynolds transport theorem.

4.54 Air enters an elbow with a uniform speed of 10 m/s as shown in Fig. P4.54. At the exit of the elbow the velocity profile is not uniform. In fact, there is a region of separation or reverse flow. The fixed control volume *ABCD* coincides with the system at time $t = 0$. Make a sketch to indicate **(a)** the system at time $t = 0.01$ s and **(b)** the fluid that has entered and exited the control volume in that time period.



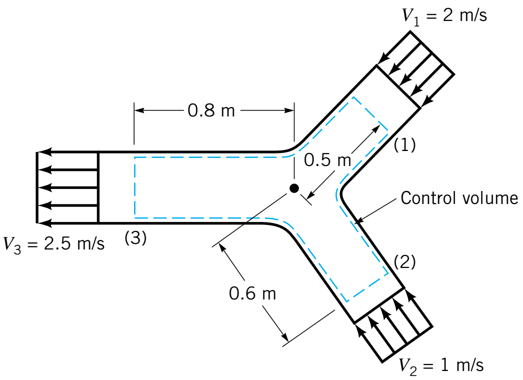
■ **FIGURE P4.54**

4.55 A layer of oil flows down a vertical plate as shown in Fig. P4.55 with a velocity of $\mathbf{V} = (V_0/h^2)(2hx - x^2)\,\hat{\mathbf{j}}$ where V_0 and h are constants. **(a)** Show that the fluid sticks to the plate and that the shear stress at the edge of the layer ($x = h$) is zero. **(b)** Determine the flowrate across surface *AB*. Assume the width of the plate is b. (*Note:* The velocity profile for laminar flow in a pipe has a similar shape. See **Video V6.6**.)



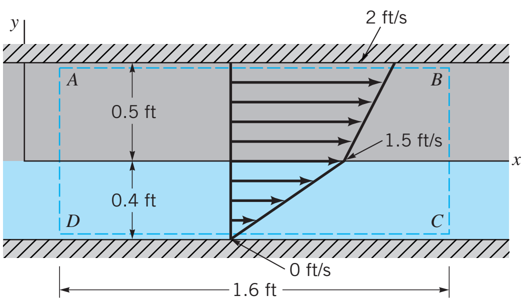
■ **FIGURE P4.55**

4.56 Water flows in the branching pipe shown in Fig. P4.56 with uniform velocity at each inlet and outlet. The fixed control volume indicated coincides with the system at time $t = 20$ s. Make a sketch to indicate **(a)** the boundary of the system at time $t = 20.2$ s, **(b)** the fluid that left the control volume during that 0.2-s interval, and **(c)** the fluid that entered the control volume during that time interval.



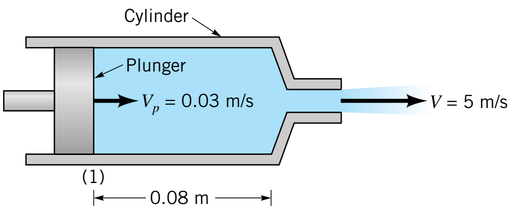
■ **FIGURE P4.56**

4.57 Two liquids with different densities and viscosities fill the gap between parallel plates as shown in Fig. P4.57. The bottom plate is fixed; the top plate moves with a speed of 2 ft/s. The velocity profile consists of two linear segements as indicated. The fixed control volume *ABCD* coincides with the system at time $t = 0$. Make a sketch to indicate **(a)** the system at time $t = 0.1$ s and **(b)** the fluid that has entered and exited the control volume in that time period.



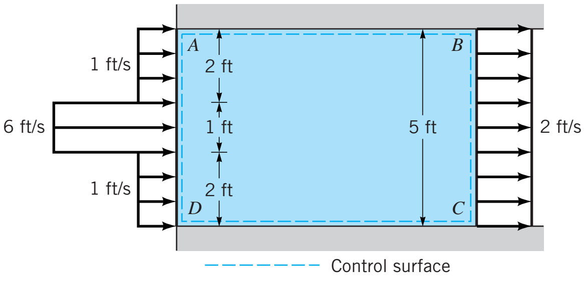
■ **FIGURE P4.57**

4.58 Water is squirted from a syringe with a speed of $V = 5$ m/s by pushing in the plunger with a speed of $V_p = 0.03$ m/s as shown in Fig. P4.58. The surface of the deforming control volume consists of the sides and end of the cylinder and the end of the plunger. The system consists of the water in the syringe at $t = 0$ when the plunger is at section (1) as shown. Make a sketch to indicate the control surface and the system when $t = 0.5$ s.



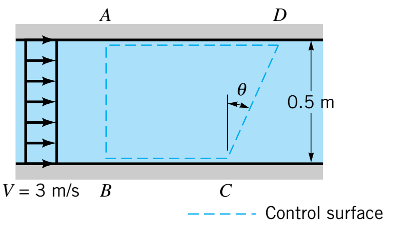
■ **FIGURE P4.58**

4.59 Water enters a 5-ft-wide, 1-ft-deep channel as shown in Fig. P4.59. Across the inlet the water velocity is 6 ft/s in the center portion of the channel and 1 ft/s in the remainder of it. Farther downstream the water flows at a uniform 2 ft/s velocity across the entire channel. The fixed control volume *ABCD* coincides with the system at time $t = 0$. Make a sketch to indicate **(a)** the system at time $t = 0.5$ s and **(b)** the fluid that has entered and exited the control volume in that time period.



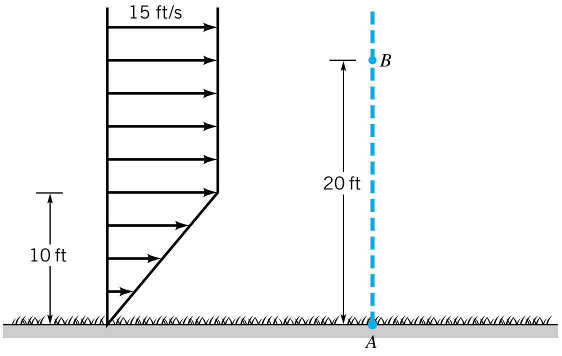
■ **FIGURE P4.59**

4.60 Water flows through the 2-m-wide rectangular channel shown in Fig. P4.60 with a uniform velocity of 3 m/s. **(a)** Directly integrate Eq. 4.16 with $b = 1$ to determine the mass flowrate (kg/s) across section *CD* of the control volume. **(b)** Repeat part (a) with $b = 1/\rho$, where ρ is the density. Explain the physical interpretation of the answer to part (b).



■ **FIGURE P4.60**

4.61 The wind blows across a field with an approximate velocity profile as shown in Fig. P4.61. Use Eq. 4.16 with the parameter b equal to the velocity to determine the momentum flowrate across the vertical surface *A–B*, which is of unit depth into the paper.



■ **FIGURE P4.61**