Exercise 2.3 (30 ธ.ค 48)

1. กำหนด $y = \dfrac{(x+1)(x+2)}{(x-1)(x-2)}$ จงหา $\dfrac{dy}{dx}$

Sol^n

$$\frac{dy}{dx} = \frac{[(x-1)(x-2)]\frac{d}{dx}[(x+1)(x+2)] - [(x+1)(x+2)]\frac{d}{dx}[(x-1)(x-2)]}{[(x-1)(x-2)]^2}$$

$$= \frac{(x^2-3x+2)(2x+3) - (x^2+3x+2)(2x-3)}{(x-1)^2(x-2)^2}$$

$$= \frac{-6x^2+12}{(x^2-3x+2)^2} \quad \#$$

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2. จงหา $\dfrac{d^2p}{dq^2}$ เมื่อ $p = \left(\dfrac{q^2+3}{12q}\right)\left(\dfrac{q^4-1}{q^3}\right)$

Sol^n Let $u = \dfrac{q^2+3}{12q}$; $v = \dfrac{q^4-1}{q^3}$

$$\frac{dp}{dq} = u\frac{dv}{dq} + v\frac{du}{dq}$$

$$\frac{du}{dq} = \frac{d}{dq}\left(\frac{q^2+3}{12q}\right) = \frac{d}{dq}\left(\frac{1}{12}q + \frac{1}{4q}\right) = \frac{1}{12} - \frac{1}{4q^2}$$

$$\frac{dv}{dq} = \frac{d}{dq}\left(q - \frac{1}{q^3}\right) = 1 + \frac{3}{q^4}$$

$$\therefore \frac{dp}{dq} = \left(\frac{q^2+3}{12q}\right)\left(1+\frac{3}{q^4}\right) + \left(\frac{q^4-1}{q^3}\right)\left(\frac{1}{12} - \frac{1}{4q^2}\right) \quad \#$$

$$\frac{d^2p}{dq^2} = \frac{d}{dq}\left(u\frac{dv}{dq} + v\frac{du}{dq}\right)$$

$$= \frac{dv}{dq}\left(\frac{du}{dq}\right) + u\frac{d^2v}{dq^2} + \frac{du}{dq}\left(\frac{dv}{dq}\right) + v\frac{d^2u}{dq^2}$$

$$= u\frac{d^2v}{dq^2} + 2\left(\frac{du}{dq}\right)\left(\frac{dv}{dq}\right) + v\frac{d^2u}{dq^2} \quad -(2)$$

$$\frac{d^2u}{dq^2} = \frac{d}{dq}\left(\frac{1}{12} - \frac{1}{4q^2}\right) = 0 + \frac{1}{2q^3} = \frac{1}{2q^3}$$

$$\frac{d^2v}{dq^2} = \frac{d}{dq}\left(1 + \frac{3}{q^4}\right) = 0 - \frac{12}{q^5} = \frac{-12}{q^5}$$

$$\therefore \frac{d^2p}{dq^2} = \left(\frac{q^2+3}{12q}\right)\left(\frac{-12}{q^5}\right) + 2\left(\frac{1}{12} - \frac{1}{4q^2}\right)\left(1 + \frac{3}{q^4}\right) + \left(\frac{q^4-1}{q^3}\right)\left(\frac{1}{2q^3}\right)$$

$$\frac{d^2p}{dq^2} = -\frac{(q^2+3)}{q^6} + \left(\frac{1}{6} - \frac{1}{2q^2}\right)\left(1+\frac{3}{q^4}\right) + \frac{q^4-1}{2q^6}$$

$$= \left(\frac{q^4-2q^2-7}{2q^6}\right) + \frac{(q^4+3)(q^2-3)}{6q^6}$$

$$= \frac{\cancel{3q^4} - \cancel{6q^2} - 21 + q^6 - \cancel{3q^4} + \cancel{3q^2} - 9}{6q^6} = \frac{q^6 - 3q^2 - 30}{6q^6} \quad \#$$

Exercise 2.4

1. จงหา $\dfrac{dy}{dx}$ เมื่อ $y = \dfrac{4}{\cos x} + \dfrac{1}{\tan x}$

Sol^n

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{4}{\cos x}\right) + \frac{d}{dx}\left(\frac{1}{\tan x}\right)$$

$$= \frac{d}{dx}(4\sec x) + \frac{d}{dx}(\cot x)$$

$$= 4\sec x \tan x - \csc^2 x \quad \#$$

OR.

$$\frac{dy}{dx} = -4(\cos x)^{-2}\frac{d}{dx}\cos x + (-1)(\tan x)^{-2}\frac{d}{dx}\tan x$$

$$= \frac{4\sin x}{\cos^2 x} - \frac{\sec^2 x}{\tan^2 x}$$

$$= 4\left(\frac{\sin x}{\cos x}\right)\frac{1}{\cos x} - \sec^2 x\left(\frac{\cos^2 x}{\sin^2 x}\right)$$

$$= 4\sec x \tan x - \csc^2 x \quad \#$$

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2 จงหา $\dfrac{ds}{dt}$ เมื่อ $s = \dfrac{\sin t}{1-\cos t}$

Sol^n

$$\frac{ds}{dt} = \frac{(1-\cos t)\frac{d}{dt}\sin t - (\sin t)\frac{d}{dt}(1-\cos t)}{(1-\cos t)^2}$$

$$= \frac{(\cos t)(1-\cos t) - (\sin t)(\sin t)}{(1-\cos t)^2}$$

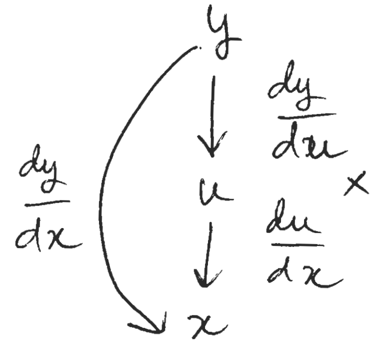
$$= \frac{\cos t - \cos^2 t - \sin^2 t}{(1-\cos t)^2} = \frac{\cos t - (\overbrace{\cos^2 t + \sin^2 t}^{=1})}{(1-\cos t)^2}$$

$$= \frac{\cos t - 1}{(1-\cos t)^2} = \frac{-(1-\cos t)}{(1-\cos t)^2} = \frac{1}{\cos t - 1} \quad \#$$

The Chain Rule

if $y = f(u)$ and $u = g(x)$ then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$



ต.ย. จงหา $\frac{dy}{dx}$ เมื่อ $y = \sin(x^2+x)$

วิธีทำ
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

ให้ $u = x^2 + x \Rightarrow \frac{du}{dx} = (2x+1)$

$$\frac{dy}{du} = \frac{d}{du}(\sin u) = \cos u = \cos(x^2+x)$$

$$\therefore \frac{dy}{dx} = [\cos(x^2+x)](2x+1)$$

$$= (2x+1)\cos(x^2+x) \#$$

กำหนดให้ $x = f(t)$
$y = g(t)$

$$\frac{dy}{dt} = \left(\frac{dy}{dx}\right)\left(\frac{dx}{dt}\right) \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

ทำนอง $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}y'$ เมื่อ $y' = \frac{dy}{dx}$

$$\therefore \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

ต.ย. จงหาสมการเส้นสัมผัสของไฮเพอร์โบลาซีกขวา ซึ่งกำหนดโดย

$$x = \sec t \quad , \quad y = \tan t \qquad -\pi/2 < t < \pi/2$$

ที่จุด $(x_0, y_0) = (\sqrt{2}, 1)$ เมื่อ $t = \pi/4$

Sol

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dt} = \sec^2 t \quad ; \quad \frac{dx}{dt} = \sec t \tan t$$

$$\frac{dy}{dx} = \frac{\sec^2 t}{(\sec t)(\tan t)} = \frac{\sec t}{\tan t} = \frac{1}{\cos t \, \frac{\sin t}{\cos t}}$$

$$= \csc t$$

$$\left.\frac{dy}{dx}\right|_{t=\pi/4} = \csc(\pi/4) = \sqrt{2} = \text{ความชัน}$$

$$(x_0, y_0) = (\sqrt{2}, 1)$$

$$y - 1 = \sqrt{2}(x - \sqrt{2}) = \sqrt{2}\,x - 2$$

$$y = \sqrt{2}\,x - 2 + 1 = \sqrt{2}\,x - 1$$

$\therefore$ Tangent line to the curve at $(\sqrt{2}, 1)$ is

$$\boxed{y = \sqrt{2}\,x - 1}$$

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ต.ย. จงหา $\frac{d^2y}{dx^2}$ เมื่อกำหนด $x = t - t^2$, $y = t - t^3$

Sol^n

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$ เมื่อ $y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$$\frac{dx}{dt} = 1 - 2t \;;\quad \frac{dy}{dt} = 1 - 3t^2$$

$$\therefore \frac{dy}{dx} = y' = \frac{1 - 3t^2}{1 - 2t}$$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1-3t^2}{1-2t}\right) = \frac{(1-2t)(-6t) - (1-3t^2)(-2)}{(1-2t)^2}$$

$$= \frac{-6t + 12t^2 + 2 - 6t^2}{(1-2t)^2} = \frac{6t^2 - 6t + 2}{(1-2t)^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{(6t^2 - 6t + 2)}{(1-2t)^2} \Big/ (1-2t)$$

$$= \frac{6t^2 - 6t + 2}{(1-2t)^3} \quad \#$$

ต.ย. จงหาความชันของเส้นสัมผัสกราฟ $y = \sin^5 x$ เมื่อ $x = \pi/3$

Sol^n

$$y = \sin^5 x$$

$$\frac{dy}{dx} = 5\sin^4 x \frac{d}{dx}(\sin x) = 5\cos x \sin^4 x$$

$$\left.\frac{dy}{dx}\right|_{x=\pi/3} = 5\cos(\pi/3)\sin^4(\pi/3) = 5\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)^4$$

$$= \frac{45}{32}$$

ต.ย. จงแสดงว่า ความชันของทุกๆเส้นสัมผัสกราฟ $y = \dfrac{1}{(1-2x)^3}$ เป็นค่าบวกเสมอ

Sol^n

$$\frac{dy}{dx} = \frac{d}{dx}[1-2x]^{-3} = -3[1-2x]^{-3-1}\frac{d(1-2x)}{dx}$$

$$= -3(1-2x)^{4}(-2) = \frac{6}{(1-2x)^4}$$

ตัวส่วนยกกำลังเลขคู่ จึงเป็นบวกเสมอ #

Exercise 2.5

1. จงหา $\dfrac{dr}{d\theta}$ เมื่อ $r = \sec\sqrt{\theta}\,\tan\left(\dfrac{1}{\theta}\right)$

Sol^n

$$\frac{dr}{d\theta} = \left[\tan\left(\frac{1}{\theta}\right)\right]\frac{d}{d\theta}\sec\sqrt{\theta} + (\sec\sqrt{\theta})\frac{d}{d\theta}\tan\left(\frac{1}{\theta}\right)$$

$$= \tan\left(\frac{1}{\theta}\right)\sec\sqrt{\theta}\tan\sqrt{\theta}\,\frac{d\sqrt{\theta}}{d\theta}$$

$$+ \sec\sqrt{\theta}\sec^2\left(\frac{1}{\theta}\right)\frac{d(1/\theta)}{d\theta}$$

$$= \frac{1}{2\sqrt{\theta}}\tan\left(\frac{1}{\theta}\right)\sec\sqrt{\theta}\tan\sqrt{\theta} - \frac{1}{\theta^2}\sec\sqrt{\theta}\sec^2\left(\frac{1}{\theta}\right)$$

ans

2 จงหา $\dfrac{d^2y}{dx^2}$ เมื่อ $x = \sec^2 t - 1,\ y = \tan t$

พร้อมทั้งหาสมการของเส้นสัมผัสกราฟ เมื่อ $t = -\pi/4$

Sol^n

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sec^2 t}{2\sec t(\sec t)(\tan t)} = \frac{1}{2\tan t}$$

$$\left.\frac{dy}{dx}\right|_{t=-\pi/4} = \frac{1}{2\tan(-\pi/4)} = -\frac{1}{2}$$

$$(x_0, y_0) = \left(\sec^2(-\pi/4) - 1,\ \tan(-\pi/4)\right) = \left(-\frac{1}{2}, -1\right)$$

$$y + 1 = -\frac{1}{2}\left(x + \frac{1}{2}\right) \Rightarrow 2y + 2 = -x - \frac{1}{2}$$

$$4y + 4 = -2x - 1 \Rightarrow \boxed{2x + 4y + 5 = 0}$$

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$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1}{2\tan t}\right) = \frac{1}{2}\frac{d}{dt}(\cot t)$$

$$= -\frac{1}{2}\csc^2 t$$

$$\frac{d^2y}{dx^2} = \frac{-\frac{1}{2}\csc^2 t}{2\sec^2 t(\tan t)} = -\frac{1}{4}\frac{\cot^2 t}{\tan t} = -\frac{1}{4}\cot^3 t \quad \#$$